

# A commutative $\mathbb{P}^1$ -spectrum representing motivic cohomology over Dedekind domains I

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## Abstract

We construct a motivic Eilenberg-MacLane spectrum with a highly structured multiplication over smooth schemes over Dedekind domains which represents Levine's motivic cohomology. The latter is defined via Bloch's cycle complexes. Our method is by gluing  $p$ -completed and rational parts along an arithmetic square. Hereby the finite coefficient spectra are obtained by truncated étale sheaves (relying on the now proven Bloch-Kato conjecture) and a variant of Geisser's version of syntomic cohomology, and the rational spectra are the ones which represent Beilinson motivic cohomology.

As application the arithmetic motivic cohomology groups can be realized as Ext-groups in a triangulated category of Tate sheaves with integral coefficients. These can be modelled as representations of derived fundamental groups.

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## 1 Introduction

This is the first part of a two-part series on the construction of a motivic Eilenberg-MacLane spectrum in mixed characteristic. The main properties of this spectrum are that it represents Bloch-Levine’s motivic cohomology of smooth schemes over Dedekind domains (with perfect residue fields (which is an assumption which is of purely technical nature and will probably be removed)) and that it is an  $E_\infty$ -ring spectrum.

The latter property makes it possible to consider the category of highly structured modules over the spectrum, thus defining a triangulated category of motivic sheaves over smooth schemes over Dedekind domains whose Ext-groups compute motivic cohomology.

The outline of this first part is as follows. In section 3 we define motivic complexes over small sites and describe their main properties, most notably the localization sequence due to Levine (Theorem 3.1).

In section 4 an  $E_\infty$ -spectrum is constructed with the main property that it represents motivic cohomology with finite coefficients (which follows from Corollary 4.1.2) and is rationally isomorphic to the Beilinson spectrum.

For the definition we use an arithmetic square, i.e. we first define  $p$ -completed spectra for all prime numbers  $p$  and glue their product along the rationalization of this product to the Beilinson spectrum (Definition 4.4).

The spectra with finite  $p$ -power coefficients which define the  $p$ -completed parts are constructed using truncated étale sheaves outside characteristic  $p$  and logarithmic de Rham-Witt sheaves at characteristic  $p$ .

In section 5 we define a second motivic spectrum which by definition represents integral motivic cohomology. To do that we introduce a strictification process for Bloch-Levine’s cycle complexes to get a strict presheaf on smooth schemes over a Dedekind

domain. Hereby we rely heavily on a moving Lemma due to Levine (Theorem 5.8). Using a localization sequence for the pair  $(\mathbb{A}^1, \mathbb{G}_m)$  we obtain bonding maps arranging the motivic complexes into a  $\mathbb{G}_m$ -spectrum (see section 5.3). This section also contains the construction of an étale cycle class map (inspired by the construction in [11]) which is compatible with certain localization sequences (Proposition 5.2.3).

In the second part of this series we will show that the two motivic spectra we defined here are in fact isomorphic relying on the properties of the motivic complexes shown here. This will show that also the first spectrum represents motivic cohomology with integral coefficients.

We also will show that the exceptional inverse image of our spectrum along the inclusion of a closed point in the Dedekind scheme yields the appropriately shifted and twisted motivic Eilenberg-MacLane spectrum over the residue field.

Our motivic Eilenberg-MacLane spectrum will be strongly periodizable in the sense of [19]. This will show that geometric mixed Tate sheaves with integral coefficients over a Dedekind domain can be modelled as representations of an affine derived group scheme along the lines of [18].

The main open question is the computation of the inverse image of our spectrum with respect to the inclusion of a closed point in the Dedekind scheme (which conjecturally should be the motivic Eilenberg-MacLane spectrum over the residue field).

It should be possible to generalize our strictification process in section 5 to define a homotopy coniveau tower over Dedekind domains as in [13]. We will come back to this question in future work.

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## 2 Preliminaries and Notation

For a site  $\mathcal{S}$  and a category  $\mathcal{C}$  we denote by  $\mathrm{Sh}(\mathcal{S}, \mathcal{C})$  the category of sheaves on  $\mathcal{S}$  with values in  $\mathcal{C}$ . If  $R$  is a commutative ring we set  $\mathrm{Sh}(\mathcal{S}, R) := \mathrm{Sh}(\mathcal{S}, \mathrm{Mod}_R)$ , where  $\mathrm{Mod}_R$  denotes the category of  $R$ -modules.

For a Noetherian separated base scheme  $S$  of finite Krull dimension we denote by  $\mathrm{Sch}_S$  the category of separated schemes of finite type over  $S$  and by  $\mathrm{Sm}_S$  the full subcategory of  $\mathrm{Sch}_S$  of smooth schemes over  $S$ .

For  $t \in \{Zar, Nis, \acute{e}t\}$  we denote by  $\mathrm{Sm}_{S,t}$  the site  $\mathrm{Sm}_S$  equipped with the topology  $t$ .

For  $S$  and  $t$  as above we denote by  $S_t$  the site consisting of the full subcategory of  $\mathrm{Sm}_S$  of étale schemes over  $S$  equipped with the topology  $t$ .

If  $m$  is invertible on  $S$  we write  $\mathbb{Z}/m(r)^S$  for the sheaf  $\mu_m^{\otimes r}$  on  $S_{\acute{e}t}$ . If it is clear from the context we also write  $\mathbb{Z}/m(r)$ .

We let  $\epsilon: \mathrm{Sm}_{S,\acute{e}t} \rightarrow \mathrm{Sm}_{S,Zar}$  and  $\epsilon: S_{\acute{e}t} \rightarrow S_{Zar}$  be the canonical maps of sites.

If  $X$  is a presheaf of sets on  $\mathrm{Sm}_S$  we let  $R[X]_t$  be the sheaf of  $R$ -modules on  $\mathrm{Sm}_{S,t}$  freely generated by  $X$ . If  $Y \hookrightarrow X$  is a monomorphism we let  $R[X, Y]_t := R[X]_t / R[Y]_t$ .

For the rest of the paper we fix a Dedekind domain  $D$  of mixed characteristic and set  $S := \mathrm{Spec}(D)$ . For a prime  $p$  we let  $S[\frac{1}{p}] := \mathrm{Spec}(D[\frac{1}{p}])$  and  $Z_p \subset S$  the closed complement of  $S[\frac{1}{p}]$  with the reduced scheme structure. Then  $Z_p$  is a finite union of spectra of fields of characteristic  $p$ .

For  $S'$  the spectrum of a Dedekind domain we let  $\mathrm{Sm}'_{S'}$  be the full subcategory of  $\mathrm{Sch}_{S'}$  of schemes  $X$  over  $S'$  such that each connected component of  $X$  is either smooth over  $S'$  or smooth over a closed point of  $S'$ .

For an  $\mathbb{F}_p$ -scheme  $Y$  we let  $W_n \Omega_Y^\bullet$  be the De Rham-Witt complex of  $Y$ . It is a complex of sheaves on  $Y_{\acute{e}t}$  with a multiplication. These complexes assemble to a complex of sheaves on the category of all  $\mathbb{F}_p$ -schemes. There are canonical epimorphisms  $W_{n+1} \Omega_Y^\bullet \twoheadrightarrow W_n \Omega_Y^\bullet$  respecting the multiplication.

For  $Y$  as above let  $\mathrm{dlog}: \mathcal{O}_Y^* \rightarrow W_n \Omega_Y^1$  be defined by  $x \mapsto \frac{dx}{x}$ , where  $\underline{x} = (x, 0, 0, \dots)$  is the Teichmüller representative of  $x$ .

The logarithmic De Rham-Witt sheaf  $W_n \Omega_{Y,\log}^r$  is defined to be the subsheaf of  $W_n \Omega_Y^r$  generated étale locally by sections of the form  $\mathrm{dlog} x_1 \dots \mathrm{dlog} x_r$ . Also  $W_n \Omega_{Y,\log}^0$  is the constant sheaf on  $\mathbb{Z}/p^n$ .

These sheaves assemble to a subcomplex  $W_n \Omega_{Y,\log}^\bullet$  of  $W_n \Omega_Y^\bullet$ .

The  $W_n \Omega_{Y,\log}^r$  assemble to a sheaf  $\nu_n^r$  on the category of all  $\mathbb{F}_p$ -schemes. We set  $\nu_n^r = 0$  for  $r < 0$ . There are natural epimorphisms  $\nu_{n+1}^r \twoheadrightarrow \nu_n^r$ .

We will also denote restrictions of  $\nu_n^r$  to certain sites, e.g. to  $Y_{Zar}$  or  $\mathrm{Sm}_{k,t}$ ,  $k$  some field of characteristic  $p$ , by  $\nu_n^r$ .

If  $\mathcal{A}$  is an abelian category we denote by  $\mathrm{D}(\mathcal{A})$  its derived category. We denote by  $\mathrm{D}^{\mathbb{A}^1}(\mathrm{Sh}(\mathrm{Sm}_{S,t}, \mathbb{Z}))$  the  $\mathbb{A}^1$ -localization of  $\mathrm{D}(\mathrm{Sh}(\mathrm{Sm}_{S,t}, \mathbb{Z}))$ .

We let  $\mathrm{SH}(S)$  be the stable motivic homotopy category and  $\mathcal{H}_\bullet(S)$  the pointed  $\mathbb{A}^1$ -homotopy category of  $S$ .

We sometimes use the same notation for a (derived) push forward or pullback between sheaf categories corresponding to sites induced by a scheme morphism. The

precise sites which are used can always be read off from the category which the objects to which the functors are applied belong to.

$E_\infty$ -structures are understood with respect to (the image of) the linear isometries operad.

### 3 Motivic complexes I

Let  $S'$  be the spectrum of a Dedekind domain. For  $X \in \mathrm{Sm}'_{S'}$  and  $r \geq 0$  we denote by  $\mathcal{M}^X(r) \in \mathrm{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}))$  Levine's cycle complex. A representative is the complex with  $z^r(-, 2r - i)$  in cohomological degree  $i$ , see [5, §3], [12]. For  $r < 0$  we set  $\mathcal{M}^X(r) = 0$ . When it is clear from the context which  $X$  is meant we also write  $\mathcal{M}(r)$ . We also write  $\mathcal{M}_{\mathrm{et}}^X(r)$  for  $\epsilon^* \mathcal{M}^X(r)$  and  $\mathcal{M}^X(r)/m$  for  $M^X(r) \otimes^{\mathbb{L}} \mathbb{Z}/m$ .

**Theorem 3.1:** *(Levine) Let  $i: Z \rightarrow X$  be a closed inclusion in  $\mathrm{Sm}'_{S'}$  of codimension  $c$  and  $j: U \rightarrow X$  the complementary open inclusion. Then there is an exact triangle in  $\mathrm{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}))$*

$$\mathbb{R}i_* \mathcal{M}^Z(r - c)[-2c] \rightarrow \mathcal{M}^X(r) \rightarrow \mathbb{R}j_* \mathcal{M}^U(r) \rightarrow \mathbb{R}i_* \mathcal{M}^Z(r - c)[-2c + 1]. \quad (1)$$

*Proof.* This is [12, Theorem 1.7]. □

**Corollary 3.2:** *Let  $i: Z \rightarrow X$  be a closed inclusion in  $\mathrm{Sm}'_{S'}$  of codimension  $c$ . Then there is a canonical isomorphism*

$$\mathbb{R}i^! \mathcal{M}^X(r) \cong \mathcal{M}^Z(r - c)[-2c]$$

*in  $\mathrm{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}))$ .*

**Theorem 3.3:** *For  $X \in \mathrm{Sm}'_{S'}$  we have  $\mathcal{H}^k(\mathcal{M}^X(r)) = 0$  for  $k > r$ .*

*Proof.* This is [5, Corollary 4.4]. □

**Theorem 3.4:** *Suppose  $X \in \mathrm{Sm}'_{S'}$  is of characteristic  $p$ . Then there is an isomorphism*

$$\mathcal{M}^X(r)/p^n \cong \nu_n^r[-r] \quad (2)$$

*in  $\mathrm{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}/p^n))$ .*

*Proof.* If  $X$  is smooth over a perfect field this is [6, Theorem 8.3]. The general case follows by a colimit argument (using [9, I. (1.10.1)]). □

**Corollary 3.5:** *Let  $p$  be a prime,  $X \in \mathrm{Sm}_S$  and  $\pi: X \rightarrow S$  the structure morphism. Let  $i: Z := \pi^{-1}(Z_p) \rightarrow X$  be the closed and  $j: U := \pi^{-1}(S[\frac{1}{p}]) \rightarrow X$  be the open inclusion. Then  $\mathcal{H}^k(\mathbb{R}j_*\mathcal{M}^U(r) \otimes^{\mathbb{L}} \mathbb{Z}/p^n) = 0$  for  $k > r$  and the natural map*

$$\mathcal{H}^r(\mathbb{R}j_*(\mathcal{M}^U(r)/p^n)) \rightarrow i_*\nu_n^{r-1} \quad (3)$$

*induced by the triangle (1) and the isomorphism (2) is an epimorphism.*

*Proof.* This follows from Theorem 3.3, the exactness of  $i_*$  and the long exact sequence of cohomology sheaves induced by the exact triangle (1).  $\square$

**Lemma 3.6:** *Suppose  $X \in \mathrm{Sm}'_{S'}$  is of characteristic  $p$ . Then the diagram*

$$\begin{array}{ccc} \mathcal{M}^X(r)/p^{n+1} & \xrightarrow{\cong} & \nu_{n+1}^r[-r] \\ \downarrow & & \downarrow \\ \mathcal{M}^X(r)/p^n & \xrightarrow{\cong} & \nu_n^r[-r] \end{array}$$

*in  $\mathrm{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}/p^{n+1}))$  commutes.*

Suppose  $m$  is invertible on  $X \in \mathrm{Sm}'_{S'}$ . Then there is a cycle class map

$$\mathcal{M}^X(r)/m \rightarrow \mathbb{R}\epsilon_*\mathbb{Z}/m(r).$$

For a definition see the proof of Theorem 3.12. The étale sheafification of the cycle class map is an isomorphism in  $\mathrm{D}(\mathrm{Sh}(X_{\acute{e}t}, \mathbb{Z}/m))$ , see [5, Theorem 1.2. 4.].

Let  $f: Y \rightarrow X$  be a flat morphism of schemes for which the motivic cycle complexes are defined. Then there is a flat pullback  $f^*\mathcal{M}^X(r) \rightarrow \mathcal{M}^Y(r)$ .

**Lemma 3.7:** *Let  $f: Y \rightarrow X$  be a flat morphism of schemes for which the motivic cycle complexes are defined. Suppose  $m$  is invertible on  $X$ . Then the diagram*

$$\begin{array}{ccc} f^*\mathcal{M}^X(r)/m & \longrightarrow & f^*\mathbb{R}\epsilon_*\mathbb{Z}/m(r) \\ \downarrow & & \downarrow \\ \mathcal{M}^Y(r)/m & \longrightarrow & \mathbb{R}\epsilon_*\mathbb{Z}/m(r) \end{array}$$

*commutes.*

*Proof.* This follows from the definition of the étale cycle class map.  $\square$

**Lemma 3.8:** *Let  $X \in \text{Sm}'_{S'}$  and suppose  $m$  is invertible on  $X$ . Let  $m'|m$ . Then the diagram*

$$\begin{array}{ccc} \mathcal{M}^X(r)/m & \longrightarrow & \mathbb{R}\epsilon_*\mathbb{Z}/m(r) \\ \downarrow & & \downarrow \\ \mathcal{M}^X(r)/m' & \longrightarrow & \mathbb{R}\epsilon_*\mathbb{Z}/m'(r) \end{array}$$

*in  $\text{D}(\text{Sh}(X_{\text{Zar}}, \mathbb{Z}/m))$  commutes.*

*Proof.* This follows from the definition of the étale cycle class map.  $\square$

**Theorem 3.9:** *Let  $X \in \text{Sm}'_{S'}$  and suppose  $m$  is invertible on  $X$ . Then there is an isomorphism*

$$\mathcal{M}^X(r)/m \cong \tau_{\leq r}(\mathbb{R}\epsilon_*\mathbb{Z}/m(r))$$

*in  $\text{D}(\text{Sh}(X_{\text{Zar}}, \mathbb{Z}/m))$  induced by the cycle class map.*

*Proof.* By [5, Theorem 1.2. 2.] we have

$$\mathcal{M}^X(r) \cong \tau_{\leq (r+1)}\mathbb{R}\epsilon_*\mathcal{M}_{\acute{e}t}^X(r).$$

By Theorem 3.3 it follows that  $\mathbb{R}^{r+1}\epsilon_*\mathcal{M}_{\acute{e}t}^X(r) = 0$ . Thus

$$\mathcal{M}^X(r)/m \cong \tau_{\leq r}(\mathbb{R}\epsilon_*\mathcal{M}_{\acute{e}t}^X(r)/m).$$

But by [5, Theorem 1.2. 4.] we have

$$\mathcal{M}_{\acute{e}t}^X(r)/m \cong \mathbb{Z}/m(r)$$

induced by the cycle class map (see the proof of [5, Theorem 1.2. 4.]). This shows the claim.  $\square$

**Theorem 3.10:** *Let  $i: Z \rightarrow X$  be a closed inclusion in  $\text{Sm}'_{S'}$  of codimension  $c$  and suppose  $m$  is invertible on  $X$ . Then there is a canonical isomorphism*

$$\mathbb{R}i^!\mathbb{Z}/m(r) \cong \mathbb{Z}/m(r-c)[-2c]$$

*in  $\text{D}(\text{Sh}(Z_{\acute{e}t}, \mathbb{Z}/m))$ .*

*Proof.* This is contained in [17].  $\square$

A consequence is the localization/Gysin exact triangle for étale cohomology.

**Corollary 3.11:** *Let  $i: Z \rightarrow X$  be a closed inclusion in  $\mathrm{Sm}'_{S'}$  of codimension  $c$  and  $j: U \rightarrow X$  the complementary open inclusion. Suppose  $m$  is invertible on  $X$ . Then there is an exact triangle*

$$i_* \mathbb{Z}/m(r-c)[-2c] \rightarrow \mathbb{Z}/m(r) \rightarrow \mathbb{R}j_* \mathbb{Z}/m(r) \rightarrow i_* \mathbb{Z}/m(r-c)[-2c+1]$$

in  $\mathrm{D}(\mathrm{Sh}(Z_{\acute{e}t}, \mathbb{Z}/m))$ .

*Proof.* This follows from Theorem 3.10 and the corresponding exact triangle involving  $\mathbb{R}i^! \mathbb{Z}/m(r)$ .  $\square$

**Theorem 3.12:** *Let  $i: Z \rightarrow X$  be a closed inclusion in  $\mathrm{Sm}'_{S'}$  of codimension  $c$  and suppose  $m$  is invertible on  $X$ . Then the diagram*

$$\begin{array}{ccc} \mathbb{R}i^! \mathcal{M}^X(r)/m & \xrightarrow{\cong} & \mathcal{M}^Z(r-c)/m[-2c] \\ \downarrow & & \downarrow \\ \mathbb{R}i^! \mathbb{R}\epsilon_* \mathbb{Z}/m(r) & & \\ \downarrow \cong & & \\ \mathbb{R}\epsilon_* \mathbb{R}i^! \mathbb{Z}/m(r) & \xrightarrow{\cong} & \mathbb{R}\epsilon_* \mathbb{Z}/m(r-c)[-2c] \end{array}$$

commutes.

*Proof.* Let  $U = X \setminus Z$ . For  $V \in \mathrm{Sm}'_{S'}$ , we denote by  $c^r(V, n)$  the set of cycles (closed integral subschemes) of  $V \times \Delta^n$  which intersect all  $V \times Y$  with  $Y$  a face of  $\Delta^n$  properly.

Let  $\mu_m^{\otimes r} \rightarrow \mathcal{G}$  be an injectively fibrant replacement in  $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{X, \acute{e}t}, \mathbb{Z}/m))$ .

Let  $V \in \mathrm{Sm}_X$ . For  $W$  a closed subset of  $X$  such that each irreducible component has codimension greater or equal to  $r$  set  $\mathcal{G}^W(V) := \ker(\mathcal{G}(V) \rightarrow \mathcal{G}(V \setminus W))$ .

As in [11, 12.3] there is a canonical isomorphism of  $H^{2r}(\mathcal{G}^W(V))$  with the free  $\mathbb{Z}/m$ -module on the irreducible components of  $W$  of codimension  $r$  and the map  $\tau_{\leq 2r} \mathcal{G}^W(V) \rightarrow H^{2r}(\mathcal{G}^W(V))[-2r]$  is a quasi isomorphism.

For  $V \in X_{\acute{e}t}$  denote by  $\mathcal{G}^r(V, n)$  the colimit of the  $\mathcal{G}^W(V \times \Delta^n)$  where  $W$  runs through the finite unions of elements of  $c^r(V, n)$ . The simplicial complex of  $\mathbb{Z}/m$ -modules  $\tau_{\leq 2r} \mathcal{G}^r(V, \bullet)$  augments to the simplicial abelian group  $z^r(V, \bullet)/m[-2r]$ . This augmentation is a levelwise quasi isomorphism. We denote by  $\mathcal{G}^r(V)$  the total complex associated to the double complex which is the normalized complex associated to  $\tau_{\leq 2r} \mathcal{G}^r(V, \bullet)$ . Thus



we get a quasi isomorphism  $\mathcal{G}^r(X) \rightarrow z^r(X)/m[-2r]$ . Here for  $V \in \text{Sm}'_S$ , the complex  $z^r(V)$  is defined to be the normalized complex associated to the simplicial abelian group  $z^r(V, \bullet)$ .

On the other hand for  $V \in X_{\acute{e}t}$  we have a canonical map  $\mathcal{G}^r(V, n) \rightarrow \mathcal{G}(V \times \Delta^n)$  compatible with the simplicial structure. We denote by  $\mathcal{G}'(V)$  the total complex associated to the double complex which is the normalized complex associated to  $\mathcal{G}(V \times \Delta^\bullet)$ . We have a canonical quasi isomorphism  $\mathcal{G}(V) \rightarrow \mathcal{G}'(V)$  and a canonical map  $\mathcal{G}^r(V) \rightarrow \mathcal{G}'(V)$ . The above groups and maps are functorial in  $V \in X_{\acute{e}t}$ .

Thus we get a map

$$z^r(-)/m[-2r] \cong \mathcal{G}^r \rightarrow \mathcal{G}' \cong \mathcal{G}$$

in  $\text{D}(\text{Sh}(X_{\acute{e}t}, \mathbb{Z}/m))$ . This is (the adjoint of) the cycle class map.

Denote by  $\tilde{\mathcal{G}}, \tilde{\mathcal{G}}', \tilde{\mathcal{G}}^{r-c}$  the analogous objects defined for  $Z$  instead for  $X$ , so we have a diagram

$$z^{r-c}(-)/m[-2(r-c)] \xleftarrow{\sim} \tilde{\mathcal{G}}^{r-c} \rightarrow \tilde{\mathcal{G}}' \xleftarrow{\sim} \tilde{\mathcal{G}}$$

in  $\text{Cpx}(\text{Sh}(Z_{\acute{e}t}, \mathbb{Z}/m))$ .

For  $V \in \text{Sm}_X$  set  $\mathcal{G}_Z(V) := \ker(\mathcal{G}(V) \rightarrow \mathcal{G}(V|_U))$ . Thus  $\mathcal{G}_Z \in \text{Cpx}(\text{Sh}(\text{Sm}_{X, \acute{e}t}, \mathbb{Z}/m))$  computes  $i_* \mathbb{R}i^! \mu_m^{\otimes r}$ .

There is an absolute purity isomorphism  $\mathcal{G}_Z \cong i_* \tilde{\mathcal{G}}[-2c]$  in  $\text{D}(\text{Sh}(\text{Sm}_{X, \acute{e}t}, \mathbb{Z}/m))$ . Choose a representative  $\varphi: \mathcal{G}_Z \rightarrow i_* \tilde{\mathcal{G}}[-2c]$  in  $\text{Cpx}(\text{Sh}(\text{Sm}_{X, \acute{e}t}, \mathbb{Z}/m))$  of this isomorphism. This exists since  $i_* \tilde{\mathcal{G}}[-2c]$  is injectively fibrant.

For  $V \in X_{\acute{e}t}$  denote by  $\mathcal{G}'_Z(V)$  the total complex associated to the double complex which is the normalized complex associated to  $\mathcal{G}_Z(V \times \Delta^\bullet)$ . Moreover let  $\mathcal{G}_Z^r(V, n)$  be the colimit of the  $\mathcal{G}^W(V \times \Delta^n)$  where  $W$  runs through the finite unions of elements of  $c^{r-c}(V|_Z, n)$ . Denote by  $\mathcal{G}_Z^r(V)$  the total complex associated to the double complex which is the normalized complex associated to  $\tau_{\leq 2r} \mathcal{G}_Z^r(V, \bullet)$ . Denote by  $z_Z^r(V)$  the complex  $z^{r-c}(V|_Z)$ .

Set  $\mathcal{G}_U(V) := \mathcal{G}(V|_U)$ ,  $\mathcal{G}'_U(V) := \mathcal{G}'(V|_U)$ ,  $\mathcal{G}_U^r(V) := \mathcal{G}^r(V|_U)$  and  $z_U^r(V) := z^r(V|_U)$ .

We have the diagram

$$\begin{array}{ccccccc}
i_* \tilde{\mathcal{G}}[-2c] & \xleftarrow{\sim} & \mathcal{G}_Z & \xrightarrow{\quad} & \mathcal{G} & \xrightarrow{\quad} & \mathcal{G}_U \\
\downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
i_* \tilde{\mathcal{G}}'[-2c] & \xleftarrow{\sim} & \mathcal{G}'_Z & \xrightarrow{\quad} & \mathcal{G}' & \xrightarrow{\quad} & \mathcal{G}'_U \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
i_* \tilde{\mathcal{G}}^{r-c}[-2c] & \xleftarrow{\sim} & \mathcal{G}^r_Z & \xrightarrow{\quad} & \mathcal{G}^r & \xrightarrow{\quad} & \mathcal{G}^r_U \\
\downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
i_* z^{r-c}(-)[-2r] & \xleftarrow{\cong} & z^r_Z(-)[-2r] & \longrightarrow & z^r(-)[-2r] & \longrightarrow & z^r_U(-)[-2r].
\end{array}$$

The upper three left most horizontal maps are induced by  $\varphi$ . The lower left square commutes by the naturality of the purity maps in étale cohomology. All other squares commute by construction. The last two arrows in each horizontal line compose to 0 and constitute an exact triangle, thus the second vertical line computes  $i_* \mathbb{R}i^!$  of the third vertical line. The claim follows.  $\square$

**Corollary 3.13:** *Let  $i: Z \rightarrow X$  be a closed inclusion in  $\mathrm{Sm}'_{S'}$  of codimension  $c$  and  $j: U \rightarrow X$  the complementary open inclusion. Suppose  $m$  is invertible on  $X$ . Then the diagram*

$$\begin{array}{ccccccc}
i_* \mathcal{M}^Z(r-c)/m[-2c] & \longrightarrow & \mathcal{M}^X(r)/m & \longrightarrow & \mathbb{R}j_* \mathcal{M}^U(r)/m & \longrightarrow & i_* \mathcal{M}^Z(r-c)/m[-2c+1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
i_* \mathbb{R}\epsilon_* \mathbb{Z}/m(r-c)[-2c] & & & & \mathbb{R}j_* \mathbb{R}\epsilon_* \mathbb{Z}/m(r) & & i_* \mathbb{R}\epsilon_* \mathbb{Z}/m(r-c)[-2c+1] \\
\downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \cong \\
\mathbb{R}\epsilon_* i_* \mathbb{Z}/m(r-c)[-2c] & \longrightarrow & \mathbb{R}\epsilon_* \mathbb{Z}/m(r) & \longrightarrow & \mathbb{R}\epsilon_* \mathbb{R}j_* \mathbb{Z}/m(r) & \longrightarrow & \mathbb{R}\epsilon_* i_* \mathbb{Z}/m(r-c)[-2c+1]
\end{array}$$

*commutes.*

*Proof.* The diagram

$$\begin{array}{ccccccc}
i_* \mathbb{R}i^! \mathcal{M}^X(r)/m & \longrightarrow & \mathcal{M}^X(r)/m & \longrightarrow & \mathbb{R}j_* \mathcal{M}^U(r)/m & \longrightarrow & i_* \mathbb{R}i^! \mathcal{M}^X(r)/m[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
i_* \mathbb{R}i^! \mathbb{R}\epsilon_* \mathbb{Z}/m & & & & & & i_* \mathbb{R}i^! \mathbb{R}\epsilon_* \mathbb{Z}/m[1] \\
\downarrow \cong & & & & & & \downarrow \cong \\
i_* \mathbb{R}\epsilon_* \mathbb{R}i^! \mathbb{Z}/m(r) & & & & \mathbb{R}j_* \mathbb{R}\epsilon_* \mathbb{Z}/m(r) & & i_* \mathbb{R}\epsilon_* \mathbb{R}i^! \mathbb{Z}/m(r)[1] \\
\downarrow \cong & & & & \downarrow \cong & & \downarrow \cong \\
\mathbb{R}\epsilon_* i_* \mathbb{R}i^! \mathbb{Z}/m(r) & \longrightarrow & \mathbb{R}\epsilon_* \mathbb{Z}/m(r) & \longrightarrow & \mathbb{R}\epsilon_* \mathbb{R}j_* \mathbb{Z}/m(r) & \longrightarrow & \mathbb{R}\epsilon_* i_* \mathbb{R}i^! \mathbb{Z}/m(r)[1]
\end{array}$$

commutes. Thus the claim follows from Theorem 3.12.  $\square$

**Theorem 3.14:** *Let  $X \in \mathrm{Sm}'_{S'}$ . Let  $q: \mathbb{A}_X^1 \rightarrow X$  be the projection. Then the canonical map*

$$\mathcal{M}^X(r) \rightarrow \mathbb{R}q_* \mathcal{M}^{\mathbb{A}_X^1}(r)$$

*is an isomorphism in  $\mathrm{D}(\mathrm{Sh}(X_{\mathrm{Zar}}, \mathbb{Z}))$ .*

*Proof.* This is [5, Corollary 3.5].  $\square$

## 4 The construction

### 4.1 The $p$ -parts

#### 4.1.1 Finite coefficients

We fix a prime  $p$  and set  $U := S[\frac{1}{p}]$ ,  $Z := Z_p$ ,  $i: Z \hookrightarrow S$  the closed and  $j: U \hookrightarrow S$  the open inclusion.

For a scheme  $X$  for which the motivic complexes are defined we set  $\mathcal{M}_n^X(r) := \mathcal{M}^X(r)/p^n$ .

For  $n \geq 1$  and  $r \in \mathbb{Z}$  let  $L_n(r) := \mu_{p^n}^{\otimes r}$  viewed as sheaf of  $\mathbb{Z}/p^n$ -modules on  $\mathrm{Sm}_{U, \acute{e}t}$ .

The pullback  $j^{-1}: \mathrm{Sm}_S \rightarrow \mathrm{Sm}_U$ ,  $X \mapsto X \times_S U$ , induces a push forward

$$j_*: \mathrm{Sh}(\mathrm{Sm}_{U, \mathrm{Zar}}, \mathbb{Z}/p^n) \rightarrow \mathrm{Sh}(\mathrm{Sm}_{S, \mathrm{Zar}}, \mathbb{Z}/p^n)$$

(we suppress the dependence on  $n$  of the functor  $j_*$ ). The same is true for étale sheaves.

Similarly, we have the pullback  $i^{-1}: \mathrm{Sm}_S \rightarrow \mathrm{Sm}_Z$ ,  $X \mapsto X \times_S Z$ , inducing also a push forward on sheaf categories.

Let  $QL_n(1) \rightarrow L_n(1)$  be a cofibrant replacement in  $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{U,\acute{e}t}, \mathbb{Z}/p^n))$  (the latter category is equipped with the local projective model structure) and let  $QL_n(1) \rightarrow RQL_n(1)$  be a fibrant replacement via a cofibration. Thus  $\mathcal{T} := RQL_n(1)[1]$  is both fibrant and cofibrant.

Recall the decomposition

$$\mathbb{R}\mathrm{Hom}_{\mathrm{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\acute{e}t}, \mathbb{Z}/p^n))}(\mathbb{G}_{m,U}, L_n(1)[1]) = L_n(1)[1] \oplus L_n(0). \quad (4)$$

The first summand splits off because the projection  $\mathbb{G}_{m,U} \rightarrow U$  has the section  $\{1\}$ .

To define the isomorphism of the remaining summand with  $L_n(0)$  we use the Gysin sequence for the situation

$$\mathbb{G}_{m,U} \hookrightarrow \mathbb{A}_S^1 \leftarrow \{1\}.$$

Let  $\iota: \mathbb{Z}/p^n[\mathbb{G}_{m,U}, \{1\}]_{\acute{e}t} \rightarrow \mathcal{T}$  be a map which classifies the canonical element  $1 \in H_{\acute{e}t}^1(\mathbb{G}_{m,U}, L_n(1))$  under the above decomposition. Note that  $\mathbb{Z}/p^n[\mathbb{G}_{m,U}, \{1\}]_{\acute{e}t}$  is cofibrant.

**Remark 4.1:** *The map  $\mathbb{Z}/p^n[\mathbb{G}_{m,U}]_{\acute{e}t} \rightarrow \mathcal{T}$  induced by  $\iota$  represents the map induced by the last map of the exact triangle*

$$L_n(1) \rightarrow \mathbb{G}_{m,U} \xrightarrow{p^n} \mathbb{G}_{m,U} \rightarrow L_n(1)[1]$$

*in  $\mathrm{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\acute{e}t}, \mathbb{Z}))$ . This follows from the construction of the Gysin isomorphism.*

We get a map

$$\mathrm{Sym}(\iota): \mathrm{Sym}(\mathbb{Z}/p^n[\mathbb{G}_{m,U}, \{1\}]_{\acute{e}t}) \rightarrow \mathrm{Sym}(\mathcal{T})$$

of commutative monoids in symmetric sequences in  $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{U,\acute{e}t}, \mathbb{Z}/p^n))$ , in other words  $\mathrm{Sym}(\mathcal{T})$  is a commutative monoid in the category of symmetric  $\mathbb{Z}/p^n[\mathbb{G}_{m,U}, \{1\}]_{\acute{e}t}$ -spectra  $\mathrm{Sp}_{\mathbb{G}_{m,U}}^\Sigma$ . In particular it gives rise to an  $E_\infty$ -object in  $\mathrm{Sp}_{\mathbb{G}_{m,U}}^\Sigma$ . Let  $\mathrm{Sym}(\mathcal{T}) \rightarrow R\mathrm{Sym}(\mathcal{T})$  be a fibrant resolution of  $\mathrm{Sym}(\mathcal{T})$  in  $E_\infty(\mathrm{Sp}_{\mathbb{G}_{m,U}}^\Sigma)$  (here  $E_\infty(\mathrm{Sp}_{\mathbb{G}_{m,U}}^\Sigma)$  is equipped with the transferred semi model structure, in particular  $R\mathrm{Sym}(\mathcal{T})$  is underlying levelwise fibrant for the local closed projective model structure and is therefore suitable to compute the derived push forward along  $\epsilon$ ).

**Lemma 4.2:** *The map  $\mathrm{Sym}(\mathcal{T}) \rightarrow R\mathrm{Sym}(\mathcal{T})$  is a level equivalence, i.e.  $\mathrm{Sym}(\mathcal{T})$  is an  $\Omega$ -spectrum.*

*Proof.* This follows from the fact that we have chosen the map  $\iota$  in such a way that the derived adjoints of the structure maps of  $\mathrm{Sym}(\mathcal{T})$  give rise to the isomorphism  $\underline{\mathrm{RHom}}((\mathbb{G}_{m,U}, \{1\}), L_n(r)[r]) \simeq L_n(r-1)[r-1]$ .  $\square$

Set  $A := \epsilon_*(R\mathrm{Sym}(\mathcal{T}))$ , so the spectrum  $A$  is  $R\mathrm{Sym}(\mathcal{T})$  viewed as  $E_\infty$ -algebra in  $\mathbb{Z}/p^n[\mathbb{G}_{m,U}, \{1\}]_{Zar}$ -spectra in  $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{U,Zar}, \mathbb{Z}/p^n))$ .

We denote by  $A_r$  the  $r$ -th level of  $A$ . Thus  $A_r \simeq \mathbb{R}\epsilon_* L_n(r)[r]$ .

Set  $A'_r := \tau_{\leq 0}(A_r)$ , where  $\tau_{\leq 0}$  denotes the good truncation at degree 0, i.e. the complex  $A'_r$  equals  $A_r$  in (cohomological) degrees  $< 0$ , consists of the cycles in degree 0 and is 0 in positive degree.

Thus by Theorem 3.9 there is for every  $X \in \mathrm{Sm}_U$  an isomorphism

$$A'_r|_{X_{Zar}} \cong \mathcal{M}_n^X(r)[r] \quad (5)$$

in  $\mathrm{D}(\mathrm{Sh}(X_{Zar}, \mathbb{Z}/p^n))$ , where  $A'_r|_{X_{Zar}}$  denotes the restriction of  $A'_r$  to  $X_{Zar}$ .

**Lemma 4.3:** *The complexes  $A'_r$  assemble to a  $\mathbb{Z}/p^n[\mathbb{G}_{m,U}, \{1\}]_{Zar}$ -spectrum  $A'$ . This spectrum is equipped with an  $E_\infty$ -structure together with a map of  $E_\infty$ -algebras  $A' \rightarrow A$  which is levelwise the canonical map  $A'_r \rightarrow A_r$ .*

*Proof.* This follows from the fact that the truncation  $\tau_{\leq 0}$  is right adjoint to the symmetric monoidal inclusion of (cohomologically) non-positively graded complexes into all complexes and that  $\mathbb{Z}/p^n[\mathbb{G}_{m,U}, \{1\}]_{Zar}$  lies in this subcategory of non-positively graded complexes.  $\square$

Let  $A' \rightarrow RA'$  be a fibrant resolution (as  $E_\infty$ -algebras in  $\mathbb{Z}/p^n[\mathbb{G}_{m,U}, \{1\}]_{Zar}$ -spectra).

**Proposition 4.4:** *The map  $A' \rightarrow RA'$  is a level equivalence, i.e.  $A'$  is an  $\Omega$ -spectrum.*

*Proof.* Set  $m := p^n$ . Let  $X \in \mathrm{Sm}_U$ . Let  $\tilde{i}: \{0\} \rightarrow \mathbb{A}_X^1$  be the closed,  $\tilde{j}: \mathbb{G}_{m,X} \rightarrow \mathbb{A}_X^1$  the open inclusion and  $q: \mathbb{A}_X^1 \rightarrow X$  the projection. By Corollary 3.11 we have an exact triangle

$$\tilde{i}_* \mathbb{Z}/m(r-1)[-2] \rightarrow \mathbb{Z}/m(r)^{\mathbb{A}_X^1} \rightarrow \mathbb{R}\tilde{j}_* \mathbb{Z}/m(r) \rightarrow \tilde{i}_* \mathbb{Z}/m(r-1)[-1].$$

Note

$$\mathbb{R}q_* \mathbb{R}\tilde{j}_* \mathbb{Z}/m(r) \simeq \underline{\mathrm{RHom}}_{\mathrm{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\acute{e}t}, \mathbb{Z}/m))}(\mathbb{G}_{m,U}, L_n(r))|_{X_{\acute{e}t}}$$

and that  $\mathbb{R}q_*$  applied to the last map in the sequence gives the projection to the second summand in our decomposition (4). Thus by construction of the map  $\iota$  this map also gives the inverse of the adjoint of the structure map in  $R\mathrm{Sym}(\mathcal{T})$ .

By Theorem 1 there is an exact triangle

$$\tilde{i}_* \mathcal{M}_n(r-1)[-2] \rightarrow \mathcal{M}_n^{\mathbb{A}^1_X}(r) \rightarrow \mathbb{R}\tilde{j}_* \mathcal{M}_n(r) \rightarrow \tilde{i}_* \mathcal{M}_n(r-1)[-1].$$

Hence by Theorem 3.3 the canonical map

$$\tau_{\leq r}(\mathbb{R}\tilde{j}_* \mathcal{M}_n(r)) \rightarrow \mathbb{R}\tilde{j}_* \mathcal{M}_n(r)$$

is an isomorphism. Thus in view of Theorem 3.9 the same truncation property holds for  $\mathbb{R}\tilde{j}_* \tau_{\leq r} \mathbb{R}\epsilon_* \mathbb{Z}/m(r)$ . Thus the map

$$\mathbb{R}\tilde{j}_* \tau_{\leq r} \mathbb{R}\epsilon_* \mathbb{Z}/m(r) \rightarrow \mathbb{R}\epsilon_* \tilde{i}_* \mathbb{Z}/m(r-1)[-1]$$

factors through  $\tau_{\leq r}(\mathbb{R}\epsilon_* \tilde{i}_* \mathbb{Z}/m(r-1)[-1])$ .

Moreover the map

$$\mathbb{R}\tilde{j}_* \tau_{\leq r} \mathbb{R}\epsilon_* \mathbb{Z}/m(r) \cong \tau_{\leq r} \mathbb{R}\tilde{j}_* \tau_{\leq r} \mathbb{R}\epsilon_* \mathbb{Z}/m(r) \rightarrow \tau_{\leq r} \mathbb{R}\tilde{j}_* \mathbb{R}\epsilon_* \mathbb{Z}/m(r)$$

is an isomorphism, thus we have a canonical map

$$\tau_{\leq r} \mathbb{R}\epsilon_* \mathbb{Z}/m(r)^{\mathbb{A}^1_X} \rightarrow \mathbb{R}\tilde{j}_* \tau_{\leq r} \mathbb{R}\epsilon_* \mathbb{Z}/m(r).$$

Using Corollary 3.13 these maps fit into the commutative diagram

$$\begin{array}{ccccc} \mathcal{M}_n^{\mathbb{A}^1_X}(r) & \longrightarrow & \mathbb{R}\tilde{j}_* \mathcal{M}_n(r) & \longrightarrow & \tilde{i}_* \mathcal{M}_n(r-1)[-1] \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ \tau_{\leq r} \mathbb{R}\epsilon_* \mathbb{Z}/m(r)^{\mathbb{A}^1_X} & \longrightarrow & \mathbb{R}\tilde{j}_* \tau_{\leq r} \mathbb{R}\epsilon_* \mathbb{Z}/m(r) & \longrightarrow & \tau_{\leq r}(\tilde{i}_* \mathbb{R}\epsilon_* \mathbb{Z}/m(r-1)[-1]), \end{array} \quad (6)$$

where the top row is part of the triangle given by Theorem 3.1. The composition

$$\begin{aligned} A'_{r-1}[-r]|_{X_{Zar}} &\rightarrow \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}/m[\mathbb{G}_{m,U}, \{1\}]_{Zar}, A'_r[-r])|_{X_{Zar}} \\ &\rightarrow \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}/m[\mathbb{G}_{m,U}]_{Zar}, \tau_{\leq r} \mathbb{R}\epsilon_* L_n(r))|_{X_{Zar}} \\ &\cong \mathbb{R}q_* \mathbb{R}\tilde{j}_* \tau_{\leq r} \mathbb{R}\epsilon_* \mathbb{Z}/m(r) \rightarrow \tau_{\leq r}(\mathbb{R}\epsilon_* \mathbb{Z}/m(r-1)[-1]) \cong A'_{r-1}[-r]|_{X_{Zar}} \end{aligned}$$

is the identity.

By Theorem 3.14  $\mathbb{R}q_* \mathcal{M}_n^{\mathbb{A}^1_X}(r)$  identifies with  $\mathcal{M}_n^X(r)$ , thus  $\mathbb{R}q_*$  applied to the left bottom arrow in (6) is an isomorphism to the trivial summand and  $\mathbb{R}q_*$  of the bottom row splits. Thus also  $\mathbb{R}q_*$  of the top row splits. This shows that in fact

$$\mathbb{R}q_* \tilde{i}_* \mathcal{M}_n(r-1)[-1] \cong \mathcal{M}_n^X(r-1)[-1]$$

is via the right vertical isomorphism and the right lower map in the diagram isomorphic to the non-trivial summand in  $\mathbb{R}q_*\mathbb{R}\tilde{j}_*\tau_{\leq r}\mathbb{R}\epsilon_*\mathbb{Z}/m(r)$ . Since this holds over every  $X \in \text{Sm}_U$  we are done.  $\square$

Thus  $B := j_*(RA')$  is a  $\mathbb{Z}/p^n[\mathbb{G}_{m,S}, \{1\}]_{Zar}$ -spectrum and computes also levelwise the derived push forward of  $A'$  along  $j$ . (Note that to compute the levelwise push forward we also could have used the levelwise model structure.)

By (5) for every  $X \in \text{Sm}_S$  we have

$$B_r|_{X_{Zar}} \cong \mathbb{R}(j_X)_*(\mathcal{M}_n^{X_U}(r))[r] \quad (7)$$

in  $\text{D}(\text{Sh}(X_{Zar}, \mathbb{Z}/p^n))$  (here  $X_U = X \times_S U$  and  $j_X$  denotes the inclusion  $X_U \hookrightarrow X$ ).

Thus by Corollary 3.5 the map  $B'_r := \tau_{\leq 0}B_r \rightarrow B_r$  is a quasi-isomorphism.

As in Lemma 4.3 the  $B'_r$  assemble to an  $E_\infty$ -algebra  $B'$ , and the natural map  $B' \rightarrow B$  is an equivalence.

By the following Lemma we could have used  $j_*A'$  instead of  $B$  and  $B'$ .

**Lemma 4.5:** *The natural map  $j_*A' \rightarrow B$  is an equivalence.*

*Proof.* Note first that each  $A'_r$  is fibrant in  $\text{Cpx}^{\leq 0}(\text{Sh}(\text{Sm}_{U,Zar}, \mathbb{Z}/p^n))$ , thus  $j_*A'$  is the derived push forward in  $\text{Cpx}^{\leq 0}(\text{Sh}(\text{Sm}_{S,Zar}, \mathbb{Z}/p^n))$ . But truncation commutes with derived push forward (both are right adjoints), so the claim follows from the fact that  $B' \rightarrow B$  is an equivalence.  $\square$

**Corollary 4.6:** *There is a natural isomorphism*

$$\mathbb{R}^r j_* L_n(r) \cong \epsilon^* \mathcal{H}^0(B'_r) = \mathcal{H}^0(B'_r)_{\acute{e}t}$$

in  $\text{Sh}(\text{Sm}_{S,\acute{e}t}, \mathbb{Z}/p^n)$ .

*Proof.* We have  $j_*A' = \epsilon_*\tau_{\leq 0}j_*(R\text{Sym}(\mathcal{T}))$ , thus

$$\begin{aligned} \mathbb{R}^r j_* L_n(r) &\cong \mathcal{H}^0(j_*((R\text{Sym}(\mathcal{T})))_r) = \mathcal{H}^0(\tau_{\leq 0}j_*((R\text{Sym}(\mathcal{T})))_r) \\ &= \epsilon^* \mathcal{H}^0(\epsilon_*\tau_{\leq 0}j_*((R\text{Sym}(\mathcal{T})))_r) = \epsilon^* \mathcal{H}^0(j_*A'_r) = \epsilon^* \mathcal{H}^0(B'_r). \end{aligned}$$

At the end we used Lemma 4.5.  $\square$

By (7) and Corollary 3.5 we have for every  $X \in \text{Sm}_S$  a natural epimorphism

$$s_X: \mathcal{H}^0(B'_r|_{X_{Zar}}) \twoheadrightarrow (i_X)_* \nu_n^{r-1}, \quad (8)$$

where  $i_X$  is the inclusion  $X \times_S Z \hookrightarrow X$ .

**Proposition 4.7:** *The maps  $s_X$  assemble to an epimorphism*

$$s: \mathcal{H}^0(B'_r) \twoheadrightarrow i_* \nu_n^{r-1}.$$

In order to prove this Proposition we describe the maps  $s_X$  in a way Geisser used to define his version of syntomic cohomology in [5, §1,6].

Let  $X \in \text{Sm}_S$ . We first give a construction of a map

$$b_X: (i_X)^* (\mathbb{R}^r j_* L_n(r)|_{X_{\acute{e}t}}) \rightarrow \nu_n^{r-1}$$

in  $\text{Sh}((X_Z)_{\acute{e}t}, \mathbb{Z}/p^n)$ . Over a complete discrete valuation ring of mixed characteristic such a map was constructed in [1, §(6.6)], see also [5, §6].

We fix a point  $\mathfrak{p} \in Z$  and let  $\Lambda$  be the completion of the discrete valuation ring  $D_{\mathfrak{p}}$ . Set  $T := \text{Spec}(\Lambda)$ . Let  $\eta$  be the generic point of  $T$ . Let  $X_T := X \times_S T$ , and let  $X_{\mathfrak{p}}$  be the special fiber and  $X_{\eta}$  the generic fiber of  $X_T$ .

We let  $j_{X_T}: X_{\eta} \rightarrow X_T$  and  $i_{X_T}: X_{\mathfrak{p}} \rightarrow X_T$  be the canonical inclusions.

Then the map

$$b_{X_T}: M_{n, X_T}^r := (i_{X_T})^* \mathbb{R}^r (j_{X_T})_* (\mathbb{Z}/p^n(r)) \rightarrow \nu_n^{r-1}$$

in [1, §(6.6)] is defined as follows (recall  $\mathbb{Z}/p^n(r) = \mu_{p^n}^{\otimes r}$ ):

By [1, Corollary (6.1.1)] the sheaf  $M_{n, X_T}^r$  is (étale) locally generated by symbols  $\{x_1, \dots, x_r\}$ ,  $x_i \in (i_{X_T})^* (j_{X_T})_* \mathcal{O}_{X_{\eta}}^*$  (for the definition of symbol see [1, §(1.2)]).

Then for any  $f_1, \dots, f_r \in (i_{X_T})^* \mathcal{O}_{X_T}^*$  the map  $b_{X_T}$  sends the symbol  $\{f_1, \dots, f_r\}$  to 0 and the symbol  $\{f_1, \dots, f_{r-1}, \pi\}$  ( $\pi$  a uniformizer of  $\Lambda$ ) to  $\text{dlog} \bar{f}_1 \dots \text{dlog} \bar{f}_{r-1}$ , where  $\bar{f}_i$  is the reduction of  $f_i$  to  $\mathcal{O}_{X_{\mathfrak{p}}}^*$ .

By multilinearity this characterizes  $b_{X_T}$  uniquely.

The base change morphism for the square

$$\begin{array}{ccc} X_{\eta} & \xrightarrow{f_{X_U}} & X_U \\ \downarrow j_{X_T} & & \downarrow j_X \\ X_T & \xrightarrow{f_X} & X \end{array}$$

applied to the sheaf  $\mathbb{Z}/p^n(r)$  on  $(X_U)_{\acute{e}t}$  yields

$$(f_X)^* \mathbb{R}^r (j_X)_* \mathbb{Z}/p^n(r) \rightarrow \mathbb{R}^r (j_{X_T})_* \mathbb{Z}/p^n(r)$$

(note that  $(f_{X_U})^* \mathbb{Z}/p^n(r) = \mathbb{Z}/p^n(r)$ ). Applying  $(i_{X_T})^*$  and noting that  $(i_{X_T})^* (f_X)^* = (i_{\mathfrak{p}})^*$  where  $i_{\mathfrak{p}}$  is the inclusion  $i_{\mathfrak{p}}: X_{\mathfrak{p}} \rightarrow X$  we get a map

$$(i_{\mathfrak{p}})^* \mathbb{R}^r (j_X)_* \mathbb{Z}/p^n(r) \rightarrow M_{n, X_T}^r.$$



Composing with  $b_{X_T}$  gives a map

$$(i_{\mathfrak{p}})^* \mathbb{R}^r(j_X)_* \mathbb{Z}/p^n(r) \rightarrow \nu_n^{r-1}.$$

Taking the disjoint union over all points in  $Z$  we finally get the map

$$b_X: (i_X)^* \mathbb{R}^r(j_X)_* \mathbb{Z}/p^n(r) \rightarrow \nu_n^{r-1},$$

the adjoint of which is a map

$$b'_X: \mathbb{R}^r(j_X)_* \mathbb{Z}/p^n(r) \rightarrow (i_X)_* \nu_n^{r-1}.$$

Together with the isomorphism of Corollary 4.6 we get the composition

$$s'_X: \mathcal{H}^0(B'_r)|_{X_{Zar}} \rightarrow \epsilon_* \mathcal{H}^0(B_r)_{\acute{e}t}|_{X_{Zar}} \cong \epsilon_* \mathbb{R}^r(j_X)_* \mathbb{Z}/p^n(r) \rightarrow (i_X)_* \nu_n^{r-1}$$

(by our convention  $\nu_n^{r-1}$  also denotes the logarithmic De Rham-Witt complex on  $(X_Z)_{Zar}$ ).

**Proposition 4.8:** *With the notation as above we have  $s_X = s'_X$ .*

*Proof.* We keep the local completed situation at a point  $\mathfrak{p}$  of  $Z$  from above.

We have a natural map induced by flat pullback  $(f_{X_U})^* \mathcal{M}_n^{X_U}(r) \rightarrow \mathcal{M}_n^{X_\eta}(r)$ , whence we get a base change morphism

$$f_X^* \mathbb{R}^r(j_X)_* \mathcal{M}_n^{X_U}(r) \rightarrow \mathbb{R}^r(j_{X_T})_* \mathcal{M}_n^{X_\eta}(r).$$

We get a diagram

$$\begin{array}{ccccc} f_X^* \mathbb{R}^r(j_X)_* \mathcal{M}_n^{X_U}(r) & \longrightarrow & \mathbb{R}^r(j_{X_T})_* \mathcal{M}_n^{X_\eta}(r) & \longrightarrow & \mathcal{H}^{r-1}((i_{X_T})_* \mathcal{M}_n^{X_{\mathfrak{p}}}(r-1)) \\ \downarrow & & \downarrow & & \downarrow \cong \\ f_X^* \epsilon_* \mathbb{R}^r(j_X)_* \mathbb{Z}/p^n(r) & \longrightarrow & \epsilon_* \mathbb{R}^r(j_{X_T})_* \mathbb{Z}/p^n(r) & \longrightarrow & (i_{X_T})_* \nu_n^{r-1}. \end{array}$$

The left and middle vertical maps are induced by the isomorphism of Corollary 4.6 and (7). The left lower horizontal map is induced by the transformation  $f_X^* \epsilon_* \rightarrow \epsilon_* f_X^*$ . The upper right horizontal arrow is part of the localization sequence for the motivic complexes. The lower right horizontal map is induced by  $b_{X_T}$ .

The claim of the Proposition follows from the commutativity of the outer square. Indeed, a map from the left upper corner to the right lower corner is adjoint to a map

$$(i_{\mathfrak{p}})^* \mathbb{R}^r(j_X)_* \mathcal{M}_n^{X_U}(r) = (i_{X_T})^* f^* \mathbb{R}^r(j_X)_* \mathcal{M}_n^{X_U}(r) \rightarrow \nu_n^{r-1}.$$

The assertion that the outside compositions are the same implies that the adjoints of  $s_X$  and  $s'_X$  coincide over the point  $\mathfrak{p}$ . Since this is true for all points in  $Z$  the claim follows.

The left square of the above square commutes by naturality of the cycle class map, Lemma 3.7.

So we are left to prove the commutativity of the right hand square.

Since the right lower corner is an étale sheaf we can also sheafify this square in the étale topology to test commutativity.

The resulting square is adjoint to a square

$$\begin{array}{ccc} (i_{X_T})^* \epsilon^* \mathbb{R}^r(j_{X_T})_* \mathcal{M}_n^{X_\eta}(r) & \longrightarrow & \epsilon^* \mathcal{H}^{r-1}(\mathcal{M}_n^{X_{\mathfrak{p}}}(r-1)) \\ \downarrow \cong & & \downarrow \cong \\ (i_{X_T})^* \mathbb{R}^r(j_{X_T})_* \mathbb{Z}/p^n(r) & \longrightarrow & \nu_n^{r-1} \end{array}$$

(the left vertical map is an isomorphism by Corollary 4.6). This commutativity would follow from the commutativity of the right hand square in the first diagram in the proof of [5, Theorem 1.3]. This commutativity is not explicitly stated in loc. cit., but the proof in loc. cit. that  $\kappa \circ \alpha \circ c$  is 0 shows the commutativity of our diagram:

As in loc. cit. let  $R$  be the strictly henselian local ring of a point in the closed fiber  $X_{\mathfrak{p}}$  of  $X_T$ , let  $L$  be the field of quotients of  $R$ ,  $F$  the field of quotients of  $R/\pi$ ,  $V = R_{(\pi)}$ ,  $V^h$  the henselization of  $V$  and  $L^h$  the quotient field of  $V^h$ .

We have to show the commutativity of

$$\begin{array}{ccc} H^r(R[\frac{1}{\pi}], \mathcal{M}_n(r)) & \longrightarrow & H^{r-1}(R/\pi, \mathcal{M}_n(r-1)) \\ \downarrow \cong & & \downarrow \cong \\ H_{\text{ét}}^r(R[\frac{1}{\pi}], \mathbb{Z}/p^n(r)) & \longrightarrow & \nu_n^{r-1}(R/\pi). \end{array}$$

The map  $\nu_n^{r-1}(R/\pi) \rightarrow \nu_n^{r-1}(F)$  is injective (see the proof of [5, Theorem 1.3], where it is attributed to [7, Corollary 1.6]).

Thus by the naturality of the localization sequence for motivic complexes and the fact that the  $b_{X_T}$  are sheaf maps it is enough to show commutativity of the square which one gets from the last square by replacing  $R[\frac{1}{\pi}]$  with  $L$  and  $R/\pi$  with  $F$ . But this square factors as

$$\begin{array}{ccccc} H^r(L, \mathcal{M}_n(r)) & \longrightarrow & H^r(L^h, \mathcal{M}_n(r)) & \longrightarrow & H^{r-1}(F, \mathcal{M}_n(r-1)) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H_{\text{ét}}^r(L, \mathbb{Z}/p^n(r)) & \longrightarrow & H_{\text{ét}}^r(L^h, \mathbb{Z}/p^n(r)) & \longrightarrow & \nu_n^{r-1}(F). \end{array}$$

The right upper horizontal map is induced from the localization sequence of the motivic complexes for  $V^h$ , its generic and its closed point.

The left hand square commutes by naturality of the cycle class map, and the commutativity of the right hand square is shown in the proof of [5, Theorem 1.3] in the paragraph before the last paragraph. This finishes the proof.  $\square$

We next discuss functoriality of the construction of the morphisms  $s'_X$ . So let  $g: Y \rightarrow X$  be a morphism in  $\text{Sms}$ . We still keep the local completed situation from above. We let  $g_Z$ ,  $g_T$ ,  $g_\eta$  and  $g_{\mathfrak{p}}$  be the base changes of  $g$  (over  $S$ ) to  $Z$ ,  $T$ ,  $\eta$  and  $\mathfrak{p}$ .

Consider the diagram

$$\begin{array}{ccc} Y_\eta & \xrightarrow{g_\eta} & X_\eta \\ \downarrow j_{Y_T} & & \downarrow j_{X_T} \\ Y_T & \xrightarrow{g_T} & X_T \\ \uparrow i_{Y_T} & & \uparrow i_{X_T} \\ Y_{\mathfrak{p}} & \xrightarrow{g_{\mathfrak{p}}} & X_{\mathfrak{p}} \end{array}$$

A base change morphism gives us

$$(g_T)^* \mathbb{R}^r(j_{X_T})_*(\mathbb{Z}/p^n(r)) \rightarrow \mathbb{R}^r(j_{Y_T})_*(\mathbb{Z}/p^n(r)).$$

Applying  $(i_{Y_T})^*$  and using  $(i_{Y_T})^*(g_T)^* \cong (g_{\mathfrak{p}})^*(i_{X_T})^*$  gives

$$(g_{\mathfrak{p}})^* M_{n, X_T}^r \rightarrow M_{n, Y_T}^r.$$

**Lemma 4.9:** *The diagram*

$$\begin{array}{ccc} (g_{\mathfrak{p}})^* M_{n, X_T}^r & \xrightarrow{(g_{\mathfrak{p}})^*(b_{X_T})} & (g_{\mathfrak{p}})^* \nu_n^{r-1} \\ \downarrow & & \downarrow \\ M_{n, Y_T}^r & \xrightarrow{b_{Y_T}} & \nu_n^{r-1} \end{array}$$

*commutes.*

*Proof.* This follows from the definition of the morphisms  $b_{X_T}$  and  $b_{Y_T}$  in terms of symbols and the functoriality of the symbols.  $\square$

As above for  $X$  let  $f_Y$  be the map  $Y_T \rightarrow Y$ .

**Lemma 4.10:** *The diagram*

$$\begin{array}{ccc} g_T^* f_X^* \mathbb{R}^r(j_X)_* \mathbb{Z}/p^n(r) & \longrightarrow & g_T^* \mathbb{R}^r(j_{X_T})_* \mathbb{Z}/p^n(r) \\ \downarrow & & \downarrow \\ f_Y^* \mathbb{R}^r(j_Y)_* \mathbb{Z}/p^n(r) & \longrightarrow & \mathbb{R}^r(j_{Y_T})_* \mathbb{Z}/p^n(r), \end{array}$$

where all maps are induced by base change morphisms, commutes.

*Proof.* This follows by the naturality of the base change morphisms.  $\square$

**Corollary 4.11:** *The diagram*

$$\begin{array}{ccc} (g_Z)^*(i_X)^* \mathbb{R}^r(j_X)_* \mathbb{Z}/p^n(r) & \xrightarrow{(g_Z)^*(b_X)} & (g_Z)^* \nu_n^{r-1} \\ \downarrow & & \downarrow \\ (i_Y)^* \mathbb{R}^r(j_Y)_* \mathbb{Z}/p^n(r) & \xrightarrow{b_Y} & \nu_n^{r-1}, \end{array}$$

where the left vertical map is induced by a base change morphism, commutes.

*Proof.* This follows by combining Lemmas 4.9 and 4.10.  $\square$

**Corollary 4.12:** *The diagram*

$$\begin{array}{ccc} g^* \mathbb{R}^r(j_X)_* \mathbb{Z}/p^n(r) & \xrightarrow{g^*(b'_X)} & g^*(i_X)_* \nu_n^{r-1} \\ \downarrow & & \downarrow \cong \\ & & (i_Y)_*(g_Z)^* \nu_n^{r-1} \\ \downarrow & & \downarrow \\ \mathbb{R}^r(j_Y)_* \mathbb{Z}/p^n(r) & \xrightarrow{b'_Y} & (i_Y)_* \nu_n^{r-1} \end{array}$$

commutes.

*Proof.* We check that the adjoints with respect to the pair  $(i_Y)^*$ ,  $(i_Y)_*$  of the two compositions are the two compositions of Corollary 4.11. For the composition via the left lower corner this is immediate. For the other composition one uses a compatibility between adjoints and pullbacks.  $\square$

**Corollary 4.13:** *The maps  $s'_X$  assemble to a map of sheaves  $\mathcal{H}^0(B'_r) \rightarrow i_* \nu_n^{r-1}$ .*

*Proof.* This follows directly from Corollary 4.12.  $\square$

*Proof of Proposition 4.7.* The assertion follows by combining Proposition 4.8 and Corollary 4.13.  $\square$

Let  $C_r$  be the kernel of the composition

$$B'_r \twoheadrightarrow \mathcal{H}^0(B'_r) \xrightarrow{s} i_* \nu_n^{r-1}.$$

Then by construction of the maps  $s_X$  we have for any  $X \in \text{Sm}_S$  an isomorphism

$$C_r|_{X_{Zar}} \cong \mathcal{M}_n^X(r)[r] \quad (9)$$

in  $\text{D}(\text{Sh}(X_{Zar}, \mathbb{Z}/p^n))$  since both objects appear as (shifted) homotopy fibers of the map

$$\mathbb{R}(j_X)_* \mathcal{M}_n^{X_U}(r) \rightarrow \mathbb{R}(i_X)_* \mathcal{M}_n^{X_Z}(r-1)[-1].$$

This isomorphism is even uniquely determined since there are no non-trivial maps  $\mathcal{M}_n^X(r) \rightarrow i_* \nu_n^{r-1}[-r-1]$  in  $\text{D}(\text{Sh}(X_{Zar}, \mathbb{Z}/p^n))$ .

**Lemma 4.14:** *Let  $R$  be a commutative ring,  $T \in \text{Sh}(\text{Sm}_{S,Zar}, R)$  and  $E$  an  $E_\infty$ -algebra in symmetric  $T$ -spectra in  $\text{Cpx}^{\leq 0}(\text{Sh}(\text{Sm}_{S,Zar}, R))$ . Let  $E_r$  be the levels of  $E$ . Let for any  $r > 0$  an epimorphism  $\mathcal{H}^0(E_r) \twoheadrightarrow e_r$  in  $\text{Sh}(\text{Sm}_{S,Zar}, R)[\Sigma_r]$  be given. Let  $E'_r$  be the kernel of the induced map  $E_r \rightarrow e_r$  and set  $E'_0 := E_0$ . Suppose the canonical map  $\varphi: T \rightarrow E_1$  (which is the composition  $T \cong R \otimes T \xrightarrow{\text{u} \otimes \text{id}} E_0 \otimes T \rightarrow E_1$  (u abbreviates unit)) factors through  $E'_1$  and that for any  $r, r' \geq 0$  the composition in  $\text{Sh}(\text{Sm}_{S,Zar}, R)$  induced by the  $E_\infty$ -multiplication on  $E$*

$$\mathcal{H}^0(E'_r) \otimes \mathcal{H}^0(E'_{r'}) \rightarrow \mathcal{H}^0(E_r) \otimes \mathcal{H}^0(E_{r'}) \rightarrow \mathcal{H}^0(E_{r+r'}) \rightarrow e_{r+r'}$$

*(the tensor products are over  $R$ ) is the zero map. Then there is an induced structure of an  $E_\infty$ -algebra  $E'$  in symmetric  $T$ -spectra on the collection of the  $E'_r$  together with a map of  $E_\infty$ -algebras  $E' \rightarrow E$  which is levelwise the canonical map  $E'_r \rightarrow E_r$ .*

*Proof.* The condition implies that we have natural maps

$$\phi_{r,r'}: \mathcal{H}^0(E'_r) \otimes \mathcal{H}^0(E'_{r'}) \rightarrow \mathcal{H}^0(E'_{r+r'}).$$

Let  $\mathcal{O}$  be our  $E_\infty$ -operad in  $\text{Cpx}^{\leq 0}(\text{Sh}(\text{Sm}_{S,Zar}, R))$ . Note that each  $E'_r$  carries an action of  $\Sigma_r$ . The structure maps of the  $E_\infty$ -algebra in  $T$ -spectra  $E$  are maps

$$s: E_r \otimes T \rightarrow E_{r+1}$$

and

$$a: \mathcal{O}(k) \otimes E_{r_1} \otimes \cdots \otimes E_{r_k} \rightarrow E_r,$$

$r = \sum_{i=1}^k r_i$ . These are subject to certain conditions. We show that when restricting these maps to the  $E'_r$  they factor through  $E'_r$  (for the appropriate  $r$ ). Then it is clear that these new structure maps also satisfy the conditions required.

To show that the composition

$$\mathcal{O}(k) \otimes E'_{r_1} \otimes \cdots \otimes E'_{r_k} \rightarrow \mathcal{O}(k) \otimes E_{r_1} \otimes \cdots \otimes E_{r_k} \rightarrow E_r$$

factors through  $E'_r$  it is sufficient to show that the induced map on  $\mathcal{H}^0$  factors through  $\mathcal{H}^0(E'_r)$ . But since  $\mathcal{O}$  is  $E_\infty$  the map on  $\mathcal{H}^0$  is a map

$$\mathcal{H}^0(E'_{r_1}) \otimes \cdots \otimes \mathcal{H}^0(E'_{r_k}) \rightarrow \mathcal{H}^0(E_r)$$

and the conditions to be  $E_\infty$  imply that this map is an iteration of the maps  $\phi_{r',r''}$ . Thus we get the factorization.

To handle the case of the  $T$ -spectrum structure maps it is again sufficient to show that the composition

$$\psi: \mathcal{H}^0(E'_r) \otimes T \rightarrow \mathcal{H}^0(E_r) \otimes T \rightarrow \mathcal{H}^0(E_{r+1})$$

factors through  $\mathcal{H}^0(E'_{r+1})$ . But the commutativity of the diagram

$$\begin{array}{ccccc}
& & \mathcal{O}(2) \otimes E_r \otimes T & & \\
& & \downarrow \cong & & \\
& & \mathcal{O}(2) \otimes E_r \otimes R \otimes T & \longrightarrow & \mathcal{O}(1) \otimes E_r \otimes T \\
& \swarrow \text{id} \otimes \varphi & \downarrow \text{id} \otimes u \otimes \text{id} & & \downarrow a \otimes \text{id} \\
& & \mathcal{O}(2) \otimes E_r \otimes E_0 \otimes T & & \\
& \swarrow \text{id} \otimes s & \searrow a \otimes \text{id} & & \\
\mathcal{O}(2) \otimes E_r \otimes E_1 & & & & E_r \otimes T \\
& \searrow a & & \swarrow s & \\
& & E_{r+1} & & 
\end{array}$$

(the only horizontal arrow is a structure map of the operad using  $R \cong \mathcal{O}(0)$ ) implies that  $\psi$  is the composition

$$\mathcal{H}^0(E'_r) \otimes T \rightarrow \mathcal{H}^0(E'_r) \otimes \mathcal{H}^0(E'_1) \rightarrow \mathcal{H}^0(E_{r+1})$$

which factors through  $\mathcal{H}^0(E'_{r+1})$  by assumption. This finishes the proof.  $\square$

We want to apply Lemma 4.14 with  $T = \mathbb{Z}/p^n[\mathbb{G}_{m,S}, \{1\}]_{Zar}$ ,  $E = B'$  and  $e_r = i_*\nu_n^{r-1}$ . Then we have  $E'_r = C_r$ .

**Lemma 4.15:** *The  $\Sigma_r$ -action on  $\mathcal{H}^0(B'_r)$  is the sign representation.*

*Proof.* This follows from the fact that there is a zig zag of quasi-isomorphisms between  $\mathcal{T}^{\otimes r}$  and  $(L_n(1)[1])^{\otimes r}$ , and on the latter the  $\Sigma_r$ -action is strictly the sign representation.  $\square$

So if we equip  $\nu_n^{r-1}$  with the sign representation of  $\Sigma_r$  the map  $\mathcal{H}^0(B'_r) \rightarrow i_*\nu_n^{r-1}$  is  $\Sigma_r$ -equivariant.

The exact sequence

$$0 \rightarrow L_n(1) \rightarrow \mathbb{G}_{m,U} \xrightarrow{p^n} \mathbb{G}_{m,U} \rightarrow 0$$

on  $\mathrm{Sm}_{U,\acute{e}t}$  induces a boundary homomorphism

$$\beta: j_*\mathbb{G}_{m,U} \rightarrow \mathbb{R}^1 j_* L_n(1)$$

of sheaves on  $\mathrm{Sm}_{S,\acute{e}t}$ . We denote the precomposition of  $\beta$  with the canonical map  $\mathbb{G}_{m,S} \rightarrow j_*\mathbb{G}_{m,U}$  by  $\beta'$ .

**Lemma 4.16:** *The composition*

$$\mathbb{G}_{m,S} \rightarrow \mathbb{Z}/p^n[\mathbb{G}_{m,S}, \{1\}]_{Zar} \xrightarrow{\varphi} B'_1 \rightarrow \mathcal{H}^0(B'_1) \rightarrow \mathcal{H}^0(B'_1)_{\acute{e}t} \cong \mathbb{R}^1 j_* L_n(1)$$

*equals  $\beta'$ .*

*Proof.* This follows from the defining property of the map  $\iota$ .  $\square$

**Corollary 4.17:** *The composition*

$$\mathbb{G}_{m,S} \rightarrow \mathbb{Z}/p^n[\mathbb{G}_{m,S}, \{1\}]_{Zar} \xrightarrow{\varphi} B'_1 \rightarrow \mathcal{H}^0(B'_1) \rightarrow i_*\nu_n^{r-1}$$

*is the constant map to zero.*

*Proof.* This follows from Lemma 4.16, the definition of the map  $b_{X_T}$  and the definition of symbol: The symbol  $\{x\}$  for  $x$  an invertible section over a smooth scheme over  $S$  is sent to 0 via  $b_{X_T}$ .  $\square$

Thus the first condition of Lemma 4.14 about the factorization of the map  $\varphi$  is satisfied.

For the second condition we get back to our local completed situation. Let  $X \in \text{Sm}_S$ ,  $\mathfrak{p} \in Z$  and let the notation be as above. By [10, §3, top of p. 277] there is an exact sequence

$$0 \rightarrow U^0 M_n^r \rightarrow M_n^r \rightarrow \nu_n^{r-1} \rightarrow 0 \quad (10)$$

on  $(X_{\mathfrak{p}})_{\text{ét}}$ , where  $U^0 M_n^r$  is the subsheaf of  $M_n^r$  generated étale locally by symbols  $\{x_1, \dots, x_r\}$  with  $x_i \in (i_{X_T})^* \mathcal{O}_{X_T}^*$ . This follows from the exact sequence

$$0 \rightarrow U^1 M_n^r \rightarrow M_n^r \rightarrow \nu_n^r \oplus \nu_n^{r-1} \rightarrow 0$$

([1, Theorem (1.4)(i)]), where  $U^1 M_n^r$  is generated étale locally by symbols  $\{x_1, \dots, x_r\}$  with  $x_i - 1 \in \pi \cdot (i_{X_T})^* \mathcal{O}_{X_T}$ . Indeed, given an element in the kernel of  $M_n^r \rightarrow \nu_n^{r-1}$  we can first change it by symbols  $\{x_1, \dots, x_r\}$  with  $x_i \in (i_{X_T})^* \mathcal{O}_{X_T}^*$  to lie also in the kernel of the map  $M_n^r \rightarrow \nu_n^r$ , and then it lies in  $U^1 M_n^r$  which is also generated by symbols (of the indicated type).

**Lemma 4.18:** *Let  $r, r' \geq 0$ . The composition*

$$\mathcal{H}^0(C_r) \otimes \mathcal{H}^0(C_{r'}) \rightarrow \mathcal{H}^0(B'_r) \otimes \mathcal{H}^0(B'_{r'}) \rightarrow \mathcal{H}^0(B'_{r+r'}),$$

*where the second map is induced by the  $E_\infty$ -structure on  $B'$ , factors through  $\mathcal{H}^0(C_{r+r'})$ .*

*Proof.* Let  $y$  be a local section lying in kernel of  $\mathcal{H}^0(B'_r) \rightarrow \nu_n^{r-1}$ , similarly for  $y'$ . We may view  $y$  and  $y'$  as local sections of  $M_n^r$  and  $M_n^{r'}$ . They are mapped to 0 by the maps to  $\nu_n^{r-1}$  and  $\nu_n^{r'-1}$ , thus by the exact sequence (10) the sections  $y$  and  $y'$  can be written locally as linear combinations of symbols of the form  $\{x_1, \dots, x_r\}$  and  $\{x'_1, \dots, x'_{r'}\}$  with  $x_i, x'_i \in (i_{X_T})^* \mathcal{O}_{X_T}^*$ . But the product of such symbols is just the concatenated symbol  $\{x_1, \dots, x_r, x'_1, \dots, x'_{r'}\}$  which thus also lies in the kernel of the map  $M_n^{r+r'} \rightarrow \nu_n^{r+r'-1}$ . This is true over all points  $\mathfrak{p}$  of  $Z$ , so we see that  $y \otimes y'$  is sent to 0 in  $i_* \nu_n^{r+r'-1}$ .  $\square$

**Corollary 4.19:** *The collection of the  $C_r$  forms an  $E_\infty$ -algebra  $C$  in  $\mathbb{Z}/p^n[\mathbb{G}_{m,S}, \{1\}]_{\text{Zar}}$ -spectra which comes with a map of  $E_\infty$ -algebras  $C \rightarrow B'$  which is levelwise the canonical map  $C_r \rightarrow B'_r$ .*

*Proof.* This follows with Corollary 4.17 and Lemma 4.18 from Lemma 4.14.  $\square$

Thus with (9) we have arranged the motivic complexes  $\mathcal{M}_n^X(r)[r]$ ,  $r \geq 0$ , into an  $E_\infty$ -algebra in  $\mathbb{Z}/p^n[\mathbb{G}_{m,S}, \{1\}]_{\text{Zar}}$ -spectra on  $\text{Sm}_{S, \text{Zar}}$ .



**Proposition 4.20:** *The algebra  $C$  is an  $\Omega$ -spectrum.*

*Proof.* Set  $m := p^n$ . Let  $X \in \text{Sm}_S$ . Let  $\tilde{i}: \{0\} \rightarrow \mathbb{A}_X^1$  be the closed,  $\tilde{j}: \mathbb{G}_{m,X} \rightarrow \mathbb{A}_X^1$  the open inclusion and  $q: \mathbb{A}_X^1 \rightarrow X$  the projection.

Since  $\mathbb{Z}/m[\mathbb{G}_{m,S}]_{Zar} \cong \mathbb{Z}/m \oplus \mathbb{Z}/m[\mathbb{G}_{m,S}, \{1\}]_{Zar}$  we have a decomposition

$$\underline{\mathbb{R}\text{Hom}}(\mathbb{G}_{m,S}, C_r) \cong C_r \oplus \mathcal{R}.$$

By Theorem 3.1 we have an exact triangle

$$\tilde{i}_* \mathcal{M}_n(r-1)[-2] \rightarrow \mathcal{M}_n^{\mathbb{A}_X^1}(r) \rightarrow \mathbb{R}\tilde{j}_* \mathcal{M}_n(r) \rightarrow \tilde{i}_* \mathcal{M}_n(r-1)[-1].$$

The composition

$$\begin{aligned} C_r[-r]|_{X_{Zar}} &\cong \mathbb{R}q_* \mathcal{M}_n^{\mathbb{A}_X^1}(r) \rightarrow \mathbb{R}q_* \mathbb{R}\tilde{j}_* \mathcal{M}_n(r) \\ &\cong \underline{\mathbb{R}\text{Hom}}(\mathbb{G}_{m,S}, C_r[-r])|_{X_{Zar}} \cong C_r[-r]|_{X_{Zar}} \oplus \mathcal{R}[-r]|_{X_{Zar}} \rightarrow C_r[-r]|_{X_{Zar}} \end{aligned}$$

is the identity. Thus when we apply  $\mathbb{R}q_*$  to the above triangle we obtain a split triangle.

Let  $\phi: \mathcal{M}_n^X(r-1)[-1] \xrightarrow{\cong} \mathcal{R}[-r]|_{X_{Zar}}$  be the resulting isomorphism.

We are finished when we prove that the diagram

$$\begin{array}{ccc} C_{r-1}|_{X_{Zar}} & \xrightarrow{\quad} & \underline{\mathbb{R}\text{Hom}}(\mathbb{Z}/m[\mathbb{G}_{m,S}, \{1\}]_{Zar}, C_r)|_{X_{Zar}} \\ \downarrow \cong & & \downarrow \cong \\ \mathcal{M}_n^X(r-1)[r-1] & \xrightarrow[\cong]{\phi} & \mathcal{R}|_{X_{Zar}}, \end{array}$$

where the upper horizontal map is the derived adjoint of the structure map of the spectrum  $C$ , commutes. To see this it is sufficient to show that the post composition of the two compositions with the map  $\mathcal{R}|_{X_{Zar}} \rightarrow \mathbb{R}j_* \mathcal{R}'|_{X_{Zar}}$ , where  $\mathcal{R}'$  is defined to be the second summand in the decomposition  $\underline{\mathbb{R}\text{Hom}}(\mathbb{G}_{m,U}, A'_r) \cong A'_r \oplus \mathcal{R}'$ , coincide, since there are no non-trivial maps from  $C_{r-1}|_{X_{Zar}}$  to  $(i_X)_* \nu_n^{r-2}[-1]$ .

But we have a transformation of diagrams from the above diagram to the diagram

$$\begin{array}{ccc} B'_{r-1}|_{X_{Zar}} & \xrightarrow{\quad} & \underline{\mathbb{R}\text{Hom}}(\mathbb{Z}/m[\mathbb{G}_{m,S}, \{1\}]_{Zar}, B'_r)|_{X_{Zar}} \\ \downarrow \cong & & \downarrow \cong \\ \mathbb{R}(j_X)_* \mathcal{M}_n^{X_U}(r-1)[r-1] & \xrightarrow[\cong]{} & \mathbb{R}j_* \mathcal{R}'|_{X_{Zar}} \end{array} \quad (11)$$

which commutes by the arguments in the proof of Proposition 4.4. So the two prolonged compositions in question are the two compositions in diagram (11) precomposed with the map  $C_{r-1}|_{X_{Zar}} \rightarrow B'_{r-1}|_{X_{Zar}}$ , thus they coincide. This finishes the proof.  $\square$

#### 4.1.2 The $p$ -completed parts

In this section we want to arrange (variants of) the  $C$  for varying  $n$  into a compatible family, such that we can then take the (homotopy) limit of this system.

To start with write  $\mathbb{Z}/p^\bullet$  for the inverse system comprised by the commutative rings  $\mathbb{Z}/p^n$  with the obvious transition maps and  $\text{Mod}_{\mathbb{Z}/p^\bullet}$  for the category of modules over this system, i.e. the category whose objects are systems of abelian groups

$$\cdots \rightarrow M_n \rightarrow \cdots \rightarrow M_2 \rightarrow M_1$$

where each  $M_n$  is annihilated by  $p^n$ .

For a site  $\mathcal{S}$  write  $\text{Sh}(\mathcal{S}, \mathbb{Z}/p^\bullet)$  for  $\text{Sh}(\mathcal{S}, \text{Mod}_{\mathbb{Z}/p^\bullet})$ .

The system of the  $L_n(r)$  comprise a natural object  $L_\bullet(r)$  of  $\text{Sh}(\text{Sm}_{U, \acute{e}t}, \mathbb{Z}/p^\bullet)$ .

Let  $QL_\bullet(1) \rightarrow L_\bullet(1)$  be a cofibrant replacement in  $\text{Cpx}(\text{Sh}(\text{Sm}_{U, \acute{e}t}, \mathbb{Z}/p^\bullet))$  (the latter category is equipped with the inverse local projective model structure) and let  $QL_\bullet(1) \rightarrow RQL_\bullet(1)$  be a fibrant replacement via a cofibration. Thus  $\mathcal{T} := RQL_\bullet(1)[1]$  is both fibrant and cofibrant.

We claim that the maps  $\iota$  from section 4.1.1 can be arranged to a map

$$\iota: \mathbb{Z}/p^\bullet[\mathbb{G}_{m,U}, \{1\}]_{\acute{e}t} \rightarrow \mathcal{T}.$$

Indeed, suppose we have already defined  $\iota$  up to level  $n$  in such a way that on each level  $k \leq n$  the map represents the canonical element  $1 \in H_{\acute{e}t}^1(\mathbb{G}_{m,U}, L_k(1))$ . We claim that we can extend the system of maps to level  $n+1$ : Choose a representative

$$\iota': \mathbb{Z}/p^{n+1}[\mathbb{G}_{m,U}, \{1\}]_{\acute{e}t} \rightarrow \mathcal{T}_{n+1}.$$

Then the composition with the fibration  $\mathcal{T}_{n+1} \rightarrow \mathcal{T}_n$  is homotopic to the map in level  $n$ . This homotopy can be lifted giving as second endpoint the required lift.

As in section 4.1.1 the map of symmetric sequences  $\text{Sym}(\iota)$  gives rise to an  $E_\infty$ -algebra  $\text{Sym}(\mathcal{T})$  in  $\Omega\text{-}\mathbb{Z}/p^\bullet[\mathbb{G}_{m,U}, \{1\}]_{\acute{e}t}\text{-spectra}$ , and we let  $\text{Sym}(\mathcal{T}) \rightarrow R\text{Sym}(\mathcal{T})$  be a fibrant resolution.

Set  $A := \epsilon_*(R\text{Sym}(\mathcal{T}))$  and  $A' := \tau_{\leq 0}(A)$ . As in section 4.1.1  $A'$  is again an  $E_\infty$ -algebra. We set  $B := j_*A'$ . By Lemma 4.5 the algebra  $B$  computes levelwise in the  $n$ -direction the algebra which was denoted  $B'$  in section 4.1.1.

Thus we have for every  $n$  and  $r$  the epimorphism

$$s_{r,n}: \mathcal{H}^0(B_{r,n}) \rightarrow i_* \nu_n^{r-1}$$

of Proposition 4.7.

**Lemma 4.21:** *We have a commutative diagram*

$$\begin{array}{ccc} \mathcal{H}^0(B_{r,n+1}) & \xrightarrow{s_{r,n+1}} & i_* \nu_{n+1}^{r-1} \\ \downarrow & & \downarrow \\ \mathcal{H}^0(B_{r,n}) & \xrightarrow{s_{r,n}} & \nu_n^{r-1}. \end{array}$$

*Proof.* We only have to verify that a corresponding diagram involving the maps  $s'_X$  commutes. This follows by the explicit definition of the maps  $b_{X_T}$ .  $\square$

We thus get an epimorphism

$$B_r \rightarrow i_* \nu_{\bullet}^{r-1}.$$

We denote by  $C_r$  the kernel of this epimorphism.

As in section 4.1.1 we can apply a variant of Lemma 4.14 (or the Lemma levelwise in the  $n$ -direction and using functoriality) to see that the collection of the  $C_r$  gives rise to an  $E_\infty$ -algebra  $C$  together with a map of  $E_\infty$ -algebras  $C \rightarrow B$  which is levelwise (for the  $r$ -direction) the canonical map  $C_r \rightarrow B_r$ .

Let  $X \in \text{Sm}_S$ . We want to see that the canonical isomorphisms (9)

$$C_{r,n}|_{X_{\text{Zar}}} \cong \mathcal{M}_n^X(r)[r]$$

are compatible with the reductions  $\mathbb{Z}/p^{n+1} \rightarrow \mathbb{Z}/p^n$ .

First by Lemma 3.8 the diagram

$$\begin{array}{ccc} \mathcal{M}_{n+1}^{X_U}(r)[r] & \xrightarrow{\cong} & A'_{r,n+1}|_{(X_U)_{\text{Zar}}} \\ \downarrow & & \downarrow \\ \mathcal{M}_n^{X_U}(r)[r] & \xrightarrow{\cong} & A'_{r,n}|_{(X_U)_{\text{Zar}}} \end{array}$$

commutes.

This shows that if we compose the two compositions in the square

$$\begin{array}{ccc} C_{r,n+1}|_{X_{\text{Zar}}} & \xrightarrow{\cong} & \mathcal{M}_{n+1}^X(r)[r] \\ \downarrow & & \downarrow \\ C_{r,n}|_{X_{\text{Zar}}} & \xrightarrow{\cong} & \mathcal{M}_n^X(r)[r] \end{array} \tag{12}$$

with the map  $\mathcal{M}_n^X(r)[r] \rightarrow \mathbb{R}(j_X)_* \mathcal{M}_n^{X_U}(r)[r]$  the resulting two maps coincide. But  $\text{Hom}_{\mathbf{D}(\text{Sh}(X_{\text{Zar}}, \mathbb{Z}/p^{n+1}))}(C_{r,n+1}|_{X_{\text{Zar}}}, \nu_n^{r-1}[-1]) = 0$ , hence (12) commutes.

Let  $C \rightarrow C'$  be a fibrant replacement as  $E_\infty$ -algebras. Then  $D_p := \lim_n C'_{\bullet,n}$  is an  $E_\infty$ -algebra in  $\mathbb{Z}_p[\mathbb{G}_{m,S}, \{1\}]_{\text{Zar}}$ -spectra in  $\text{Cpx}(\text{Sh}(\text{Sm}_{S,\text{Zar}}, \mathbb{Z}_p))$ .

**Corollary 4.22:** *For  $X \in \text{Sm}_S$  there is an isomorphism*

$$D_{p,r}|_{X_{Zar}} \cong (\mathcal{M}^X(r))^{\wedge p}[r]$$

in  $\text{D}(\text{Sh}(X_{Zar}, \mathbb{Z}_p))$ , where  $(\mathcal{M}^X(r))^{\wedge p}$  is the  $p$ -completion of  $\mathcal{M}^X(r)$ .

*Proof.* This follows from the commutativity of (12), since the  $p$ -completion of  $\mathcal{M}^X(r)$  is the homotopy limit over all  $\mathcal{M}_n^X(r)$ .  $\square$

Next we will equip  $D_p$  with an orientation.

Denote by  $\mathcal{O}_{/U}^*$  the sheaf (in any of the considered topologies) of abelian groups represented by  $\mathbb{G}_{m,U}$  over  $\text{Sm}_U$ , let  $\mathcal{O}_{/S}^*$  be defined similarly. For  $M$  a sheaf of abelian groups we set  $M/p^n := M \otimes^{\mathbb{L}} \mathbb{Z}/p^n$ .

Using the resolution of  $\mathcal{O}_{/U}^*$  by the sheaf of meromorphic functions and the sheaf of codimension 1 cycles one sees that  $\mathbb{R}^i j_* \mathcal{O}_{/U}^* = 0$  for  $i > 0$ . Thus we have an exact triangle

$$\mathcal{O}_{/S}^* \rightarrow \mathbb{R}j_* \mathcal{O}_{/U}^* \rightarrow i_* \mathbb{Z} \rightarrow \mathcal{O}_{/S}^*[1],$$

from which we derive an exact triangle

$$\mathcal{O}_{/S}^*/p^n \rightarrow \mathbb{R}j_* \mathcal{O}_{/U}^*/p^n \rightarrow i_* \mathbb{Z}/p^n \rightarrow \mathcal{O}_{/S}^*/p^n[1]. \quad (13)$$

We have a map of exact triangles

$$\begin{array}{ccccccc} \mathcal{O}_{/U}^* & \xrightarrow{p^n} & \mathcal{O}_{/U}^* & \longrightarrow & \mathcal{O}_{/U}^*/p^n & \longrightarrow & \mathcal{O}_{/U}^*[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{R}\epsilon_* \mathcal{O}_{/U}^* & \xrightarrow{p^n} & \mathbb{R}\epsilon_* \mathcal{O}_{/U}^* & \longrightarrow & \mathbb{R}\epsilon_* L_n(1)[1] & \longrightarrow & \mathbb{R}\epsilon_* \mathcal{O}_{/U}^*[1]. \end{array}$$

The third vertical map factors uniquely through a map  $\mathcal{O}_{/U}^*/p^n \rightarrow \tau_{\leq 0}(\mathbb{R}\epsilon_* L_n(1)[1])$ . Since  $\mathbb{R}^1 \epsilon_* \mathcal{O}_{/U}^* = 0$  we see by the long exact cohomology sheaf sequences associated to these triangles that this map is an isomorphism. Note we have  $A'_{1,n} \cong \tau_{\leq 0}(\mathbb{R}\epsilon_* L_n(1)[1])$  in the derived category, and thus  $B_{1,n} \cong \mathbb{R}j_* \tau_{\leq 0}(\mathbb{R}\epsilon_* L_n(1)[1]) \cong \mathbb{R}j_* \mathcal{O}_{/U}^*/p^n$ .

We note that the diagram

$$\begin{array}{ccc} \mathbb{R}j_* \mathcal{O}_{/U}^*/p^n & \longrightarrow & i_* \mathbb{Z}/p^n \\ \downarrow \cong & & \downarrow = \\ \mathbb{R}j_* \tau_{\leq 0}(\mathbb{R}\epsilon_* L_n(1)[1]) & \longrightarrow & \mathcal{H}^0(B_{1,n}) \xrightarrow{s_{r,n}} i_* \mathbb{Z}/p^n \end{array}$$

commutes (this follows from the definition of the maps  $s_{r,n}$ , Proposition 4.8 and the definition of the maps  $s'_X$ ). Thus together with the triangle (13) we derive an isomorphism  $C_{1,n} \cong \mathcal{O}_{/S}^*/p^n$  in  $D(\mathrm{Sh}(\mathrm{Sm}_{S,Zar}, \mathbb{Z}/p^n))$ . This isomorphism is moreover unique since there are no non-trivial homomorphisms from  $\mathcal{O}_{/S}^*/p^n$  to  $i_*\mathbb{Z}/p^n[-1]$ .

We see that there is an isomorphism  $D_{p,1} \cong (\mathcal{O}_{/S}^*)^{\wedge p}$  in  $D(\mathrm{Sh}(\mathrm{Sm}_{S,Zar}, \mathbb{Z}_p))$ . We denote any such isomorphism which is compatible with the projections to  $C_{1,n}$  and  $\mathcal{O}_{/S}^*/p^n$  by  $\varphi$ .

Since  $D_p$  is an  $\Omega$ -spectrum which satisfies Nisnevich descent and is  $\mathbb{A}^1$ -local the maps  $\Sigma^{-2,-1}\Sigma_+^\infty \mathbb{P}^\infty \rightarrow D_p$  in  $\mathrm{SH}(S)$  correspond to maps

$$\mathbb{Z}[\mathbb{P}^\infty]_{Zar}[-1] \rightarrow D_{p,1}$$

in  $D(\mathrm{Sh}(\mathrm{Sm}_{S,Zar}, \mathbb{Z}))$ . We let  $\alpha: \Sigma^{-2,-1}\Sigma_+^\infty \mathbb{P}^\infty \rightarrow D_p$  correspond to  $\mathbb{Z}[\mathbb{P}^\infty]_{Zar} \rightarrow \mathcal{O}_{/S}^*[1] \rightarrow (\mathcal{O}_{/S}^*)^{\wedge p}[1] \xrightarrow{\varphi^{-1}[1]} D_{p,1}[1]$ , where the first map classifies the tautological line bundle  $\mathcal{O}(-1)$  on  $\mathbb{P}^\infty$ .

The definition of the bonding maps in  $D_p$  implies that the map  $\mathbb{Z}[\mathbb{G}_{m,S}, \{1\}]_{Zar} \rightarrow D_{p,1}$  corresponding to the unit map  $\Sigma^{-1,-1}\Sigma^\infty(\mathbb{G}_{m,S}, \{1\}) \cong \mathbf{1} \rightarrow D_p$  is the map

$$\mathbb{Z}[\mathbb{G}_{m,S}, \{1\}]_{Zar} \rightarrow \mathcal{O}_{/S}^* \rightarrow (\mathcal{O}_{/S}^*)^{\wedge p} \xrightarrow{\varphi^{-1}} D_{p,1}.$$

Note that this composition is independent of the particular choice of  $\varphi$  since we have

$$\begin{aligned} & \mathrm{Hom}_{D(\mathrm{Sh}(\mathrm{Sm}_{S,Zar}, \mathbb{Z}))}(\mathbb{Z}[\mathbb{G}_{m,S}, \{1\}]_{Zar}, (\mathcal{O}_{/S}^*)^{\wedge p}) \cong \mathbb{Z}_p \\ & \cong \lim_n \mathrm{Hom}_{D(\mathrm{Sh}(\mathrm{Sm}_{S,Zar}, \mathbb{Z}))}(\mathbb{Z}[\mathbb{G}_{m,S}, \{1\}]_{Zar}, \mathcal{O}_{/S}^*/p^n). \end{aligned}$$

Let  $\psi: (\mathbb{P}^1, \{\infty\}) \rightarrow \mathbb{G}_{m,S} \wedge S^1$  be the canonical isomorphism in  $\mathcal{H}_\bullet(S)$  and let  $c: \mathcal{H}_\bullet(S) \rightarrow D^{\mathbb{A}^1}(\mathrm{Sh}(\mathrm{Sm}_{S,Nis}, \mathbb{Z}))$  be the canonical map. Then the composition

$$\mathbb{Z}[\mathbb{P}^1, \{\infty\}] \cong c((\mathbb{P}^1, \{\infty\})) \xrightarrow{c(\psi)} c(\mathbb{G}_{m,S} \wedge S^1) \cong \mathbb{Z}[\mathbb{G}_{m,S}, \{1\}][1] \rightarrow \mathcal{O}_{/S}^*[1]$$

in  $D^{\mathbb{A}^1}(\mathrm{Sh}(\mathrm{Sm}_{S,Nis}, \mathbb{Z}))$  classifies the tautological line bundle on  $\mathbb{P}^1$ . We see from these considerations that  $\alpha$  is indeed an orientation.

**Proposition 4.23:** *The spectrum in  $\mathrm{SH}(S)$  associated with  $D_p$  is orientable.*

## 4.2 The completed part

Set  $D := \prod_p D_p$ , where the  $D_p$  are the algebras from the last section viewed as  $E_\infty$ -algebras in spectra in  $\text{Cpx}(\text{Sh}(\text{Sm}_{S, \text{Zar}}, \hat{\mathbb{Z}}))$  and the product is taken over all primes.

Then for  $X \in \text{Sm}_S$  we have

$$D_r|_{X_{\text{Zar}}} \cong (\prod_p (\mathcal{M}^X(r))^{\wedge p})[r]$$

in  $\text{D}(\text{Sh}(X_{\text{Zar}}, \hat{\mathbb{Z}}))$ .

**Corollary 4.24:** *The spectrum in  $\text{SH}(S)$  associated with  $D$  is orientable.*

*Proof.* This follows from Proposition 4.23. □

## 4.3 The rational parts

We denote by  $D_{\mathbb{Q}}$  the rationalization of  $D$  as an  $E_\infty$ -spectrum.

We denote by  $\text{H}_B$  the Beilinson spectrum over  $S$ , see [2, Definition 13.1.2]. It has a natural  $E_\infty$ -structure ([2, Corollary 13.2.6]) and is orientable ([2, 13.1.5]).

**Theorem 4.25:**  *$\text{H}_B$  is the initial  $E_\infty$ -spectrum among rational orientable  $E_\infty$ -spectra.*

*Proof.* This is [2, Corollary 13.2.15 (Rv)]. □

**Corollary 4.26:** *There is a canonical map of  $E_\infty$ -spectra  $\text{H}_B \rightarrow D_{\mathbb{Q}}$ .*

*Proof.* This follows from Corollary 4.24 and Theorem 4.25. □

## 4.4 The definition

**Definition 4.27:** *We denote by  $\text{MZ}$  the homotopy pullback in  $E_\infty$ -spectra of the diagram*

$$\begin{array}{ccc} & D & \\ & \downarrow & \\ \text{H}_B & \longrightarrow & D_{\mathbb{Q}} \end{array}$$

## 5 Motivic Complexes II

### 5.1 A strictification

In this section we enlarge the motivic complexes from section 3 to presheaves on all of  $\mathbf{Sm}_S$ . We need some preparations.

For each  $n \in \mathbb{N}$  we define a category  $\mathcal{E}_n$  together with a functor  $\varphi_n: \mathcal{E}_n \rightarrow [n]$ , where  $[n]$  is the category  $0 \rightarrow 1 \rightarrow \dots \rightarrow n$ . The objects of  $\mathcal{E}_n$  are triples  $(A, B, i)$  where  $i \in [n]$  and  $A \subset B \subset \{i, \dots, n\}$  with  $i \in A$ . There is exactly one morphism from  $(A, B, i)$  to  $(A', B', j)$  if  $i \leq j$ ,  $B \cap \{j, \dots, n\} \subset B'$  and  $A' \subset A$ , otherwise there is no such morphism. The functor  $\varphi_n$  is determined by the fact that  $(A, B, i)$  is mapped to  $i$ . We declare a map  $f$  in  $\mathcal{E}_n$  to be a weak equivalence if  $\varphi_n(f)$  is an identity.

A *category with weak equivalences* is a category  $\mathcal{C}$  together with a subcategory  $\mathcal{W}$  of  $\mathcal{C}$  such that every isomorphism in  $\mathcal{C}$  lies in  $\mathcal{W}$ . A *homotopical category* is a category with weak equivalences satisfying the two out of six property, see [4, 8.2].

For a category  $\mathcal{C}$  with weak equivalences  $\mathcal{W}$  we denote by  $L_{\mathcal{W}}^H \mathcal{C}$  its hammock localization, see [3]. If it is clear which weak equivalences are meant we also write  $L^H \mathcal{C}$ .

A morphism in  $[n]$  is defined to be a weak equivalence if it is an identity. So both  $\mathcal{E}_n$  and  $[n]$  are homotopical categories. Since  $[n]$  is the homotopy category of  $L^H([n])$  there is a natural simplicial functor  $L^H([n]) \rightarrow [n]$  which is an equivalence of simplicial categories. Composing with the natural functor  $L^H \mathcal{E}_n \rightarrow L^H([n])$  gives us the simplicial functor  $L^H \mathcal{E}_n \rightarrow [n]$ .

**Proposition 5.1:** *The natural functor  $L^H \mathcal{E}_n \rightarrow [n]$  is an equivalence of simplicial categories.*

Before giving the proof we need some preparations.

For us a *direct category* is a category with a chosen degree function, see [8, Definition 5.1.1].

**Lemma 5.2:** *Let  $I$  be a direct category and  $J \subset I$  a full subcategory such that no arrow in  $I$  has a domain which is not in  $J$  and a codomain which is in  $J$ . Let  $\mathcal{C}$  be a model category and  $D: I \rightarrow \mathcal{C}$  a cofibrant diagram for the projective model structure. Then  $D|_J$  is cofibrant in  $\mathcal{C}^J$ .*

*Proof.* The right adjoint  $r$  to the restriction functor  $\mathcal{C}^I \rightarrow \mathcal{C}^J$  is a right Quillen functor since for  $i \in I \setminus J$  we have  $r(D)(i) = *$ .  $\square$

**Lemma 5.3:** *Let  $I$  be a direct category and  $J \subset I$  a full subcategory such that no arrow in  $I$  has a domain which is not in  $J$  and a codomain which is in  $J$ . Let  $\mathcal{C}$  be a model category and  $D: I \rightarrow \mathcal{C}$  a cofibrant diagram for the projective model structure. Then the canonical map  $\text{colim}(D|_J) \rightarrow \text{colim} D$  is a cofibration.*

*Proof.* The object  $\text{colim} D$  is obtained from  $\text{colim}(D|_J)$  by succesively gluing in the  $D(i)$  for  $i \in I \setminus J$  for increasing degree of  $i$ . The domains of the attaching maps are corresponding latching spaces.  $\square$

For  $i \in [n]$  let  $\mathcal{E}_{n,i} := \varphi^{-1}(i)$  and  $\mathcal{E}_{n,\leq i}$  be the full subcategory of  $\mathcal{E}_n$  of objects  $(A, B, j)$  with  $j \leq i$ . It is easily seen that  $\mathcal{E}_{n,\leq i}$  can be given the structure of a direct category. For  $j \leq i \leq n$  let  $\mathcal{E}_{n,[j,i]} := \varphi_n^{-1}(\{j, \dots, i\})$ .

**Lemma 5.4:** *Let  $\mathcal{C}$  be a model category and  $D: \mathcal{E}_{n,\leq i} \rightarrow \mathcal{C}$  be a projectively cofibrant diagram. Then for  $k \leq j \leq i$  the restriction  $D|_{\mathcal{E}_{n,[k,j]}}$  is also cofibrant.*

*Proof.* Let  $F: \mathcal{E}_{n,[k,j]} \rightarrow \mathcal{E}_{n,\leq i}$  be the inclusion. We claim that the right adjoint to the restriction functor  $\mathcal{C}^{\mathcal{E}_{n,\leq i}} \rightarrow \mathcal{C}^{\mathcal{E}_{n,[k,j]}}$  is a right Quillen functor. This follows from the fact that for  $l < k$  and an object  $(A, B, l) \in \mathcal{E}_{n,\leq i}$  with  $A \cap [k, j] \neq \emptyset$  the category  $(A, B, l)/F$  has the initial object  $(A, B, l) \rightarrow (A \cap \{m, \dots, n\}, B \cap \{m, \dots, n\}, m)$ , where  $m = \min(A \cap [k, j])$ .  $\square$

**Lemma 5.5:** *Let  $\mathcal{C}$  be a model category and  $D: \mathcal{E}_{n,\leq i} \rightarrow \mathcal{C}$  be a projectively cofibrant diagram such that for any weak equivalence  $f$  in  $\mathcal{E}_{n,\leq i}$  the map  $D(f)$  is also a weak equivalence. Then for any  $X \in \varphi_n^{-1}(i)$  the map  $D(X) \rightarrow \text{colim} D$  is a weak equivalence.*

*Proof.* We show by descending induction on  $j$ , starting with  $j = i$ , that for any  $X \in \varphi_n^{-1}(i)$  the map  $D(X) \rightarrow \text{colim} D|_{\mathcal{E}_{n,[j,i]}}$  is a weak equivalence. For  $j = i$  this follows from the fact  $\mathcal{E}_{n,i}$  has a final object. Let the statement be true for  $0 < j+1 \leq i$  and let us show it for  $j$ . Let  $J \subset \mathcal{E}_{n,j}$  be the full subcategory on objects  $(A, B, j)$  such that  $A \cap \{j+1, \dots, i\} \neq \emptyset$ . Then we have a pushout diagram

$$\begin{array}{ccc} \text{colim} D|_J & \longrightarrow & \text{colim} D|_{\mathcal{E}_{n,j}} \\ \downarrow & & \downarrow \\ \text{colim} D|_{\mathcal{E}_{n,[j+1,i]}} & \longrightarrow & \text{colim} D|_{\mathcal{E}_{n,[j,i]}} \end{array}$$

First note that by Lemmas 5.4 and 5.2 all objects in this diagram are cofibrant. Furthermore the upper horizontal map is a cofibration by Lemma 5.3. The full subcategory



of  $J$  consisting of objects  $(A, B, j)$  with  $B = \{j, \dots, n\}$  is homotopy right cofinal in  $J$  and contractible (it has an initial object), thus  $J$  is contractible. Since the diagram  $D|_J$  is weakly equivalent to a constant diagram it follows that  $D(X) \rightarrow \operatorname{colim} D|_J$  is a weak equivalence for any  $X \in J$ , thus the upper horizontal map in the above diagram is also a weak equivalence and the induction step follows.  $\square$

**Lemma 5.6:** *Let  $\mathcal{C}$  be a model category and  $l$  the left adjoint to the pull back functor  $r: \mathcal{C}^{[n]} \rightarrow \mathcal{C}^{\mathcal{E}_n}$ . Let  $D: \mathcal{E}_n \rightarrow \mathcal{C}$  be (projectively) cofibrant such that for any weak equivalence  $f$  in  $\mathcal{E}_n$  the map  $D(f)$  is a weak equivalence. Then  $D \rightarrow r(l(D))$  is a weak equivalence.*

*Proof.* We have  $l(D)(i) = \operatorname{colim} D|_{\mathcal{E}_{n, \leq i}}$ . Thus the claim follows from Lemma 5.5.  $\square$

**Lemma 5.7:** *Let  $F: I \rightarrow J$  be an essentially surjective functor between small categories and  $\mathcal{W} \subset I$  a subcategory making  $I$  into a category with weak equivalences. Suppose  $F$  sends any map in  $\mathcal{W}$  to an isomorphism. Then the natural map  $L_{\mathcal{W}}^H I \rightarrow J$  is a weak equivalence between simplicial categories if and only if for any projectively cofibrant diagram  $D: I \rightarrow \mathbf{sSet}$  such that for any map  $f$  in  $\mathcal{W}$  the map  $D(f)$  is a weak equivalence the map  $D \rightarrow r(l(D))$  is a weak equivalence where  $l$  is the left adjoint to the pull back functor  $r: \mathbf{sSet}^J \rightarrow \mathbf{sSet}^I$ .*

*Proof.* Let  $\mathcal{C}$  be the left Bousfield localization of the model category  $\mathbf{sSet}^I$  (equipped with the projective model structure) along the maps  $\operatorname{Hom}(f, -)$  where  $f$  runs through the maps of  $\mathcal{W}$ . Then  $(L_{\mathcal{W}}^H I)^{\operatorname{op}}$  is weakly equivalent to the full simplicial subcategory of  $\mathbf{sSet}^I$  consisting of cofibrant fibrant objects which become isomorphic in  $\operatorname{Ho}(\mathbf{sSet}^I)$  to objects in the image of the composed functor  $I^{\operatorname{op}} \rightarrow \operatorname{Ho}(\mathbf{sSet}^I) \rightarrow \operatorname{Ho} \mathcal{C} \hookrightarrow \operatorname{Ho}(\mathbf{sSet}^I)$ . Similarly (but easier)  $J^{\operatorname{op}}$  is weakly equivalent to a full simplicial subcategory of  $\mathbf{sSet}^J$ . The functor  $(L_{\mathcal{W}}^H I)^{\operatorname{op}} \rightarrow J^{\operatorname{op}}$  is described via these equivalences by the restriction of the push forward  $\mathbf{sSet}^I \rightarrow \mathbf{sSet}^J$  followed by a fibrant replacement functor. The claim follows.  $\square$

*Proof of Proposition 5.1.* The claim follows from Lemmas 5.6 and 5.7.  $\square$

Let  $f: [n] \rightarrow [m]$  be a map in  $\Delta$ . We define a functor  $f_*: \mathcal{E}_n \rightarrow \mathcal{E}_m$  by setting  $f_*((A, B, i)) = (f(A), f(B), f(i))$ . One checks that this determines uniquely  $f_*$ . Thus we get a cosimplicial object  $\mathcal{E}: [n] \mapsto \mathcal{E}_n$  in the category of small categories with weak equivalences. Applying the hammock localization yields a cosimplicial simplicial category  $L^H \mathcal{E}: [n] \mapsto L^H \mathcal{E}_n$  together with a map from  $L^H \mathcal{E}$  to the standard cosimplicial simplicial category  $[\bullet]$  which is levelwise a Dwyer-Kan equivalence.

Let  $\mathcal{S}$  be the category of triples  $(A, n, (a_1, \dots, a_n))$  where  $A$  is a  $D$ -algebra such that  $\text{Spec}(A) \in \text{Sm}_S$  and  $a_1, \dots, a_n \in A$  generate  $A$  as a  $D$ -algebra. Morphisms are morphisms of  $D$ -algebras with no compatibility of the generators required. Clearly the functor  $\mathcal{S}^{\text{op}} \rightarrow \text{Sm}_S$ ,  $A \mapsto \text{Spec}(A)$ , is an equivalence onto the full subcategory of  $\text{Sm}_S$  of affine schemes.

For  $X \in \text{Sm}_S$  and  $F = \{f_1, \dots, f_n\}$  a set of closed immersions  $f_i: Z_i \hookrightarrow X$  in  $\text{Sm}_S$  we denote by  $z_F^r(X)$  the normalized chain complex associated to the simplicial abelian group  $[n] \mapsto z_F^r(X, n)$  which is the subsimplicial abelian group of  $z^r(X, \bullet)$  of cycles in good position with respect to the  $Z_i$ . We also write  $z_F^r(A)$  for  $z_F^r(\text{Spec}(A))$ . We have the following moving Lemma due to Marc Levine.

**Theorem 5.8:** *If  $X \in \text{Sm}_S$  is affine then the inclusion of  $z_F^r(X)$  into the normalized chain complex associated to  $z^r(X, \bullet)$  is a quasi isomorphism.*

*Proof.* This is [11, Theorem 4.9]. □

Let

$$(A_0, k_0, (a_{0,1}, \dots, a_{0,k_0})) \rightarrow \dots \rightarrow (A_n, k_n, (a_{n,1}, \dots, a_{n,k_n}))$$

be a chain of maps in  $\mathcal{S}$ , i.e. a  $n$ -simplex, which we denote by  $K$ , in the nerve of  $\mathcal{S}$ . Let  $i \in [n]$  and  $B \subset \{i, \dots, n\}$  with  $i \in B$ . Set  $C_{i,B} := \bigotimes_{j \in B \setminus \{i\}} A_i[T_1, \dots, T_{k_j}] \cong \bigotimes_{j \in B \setminus \{i\}} A_i[T_{j,1}, \dots, T_{j,k_j}]$ , where the tensor products are over  $A_i$ . If  $i \leq j \leq n$ ,  $B' \subset \{j, \dots, n\}$  with  $j \in B'$  and  $B \cap \{j, \dots, n\} \subset B'$  we define a map  $g_{i,B,j,B'}: C_{i,B} \rightarrow C_{j,B'}$  over the map  $A_i \rightarrow A_j$  by sending a variable  $T_{l,m}$  for  $l > j$  to the respective variable  $T_{l,m}$  and to the image of the element  $a_{l,m}$  in  $A_j$  for  $l \leq j$ . If furthermore  $j \leq k \leq n$  and  $B'' \subset \{k, \dots, n\}$  with  $k \in B''$  and  $B' \cap \{k, \dots, n\} \subset B''$  then we have

$$g_{j,B',k,B''} \circ g_{i,B,j,B'} = g_{i,B,k,B''}. \quad (14)$$

For  $t = (A, B, i) \in \mathcal{E}_n$  we let  $F_t$  be the set of closed subschemes of  $\text{Spec}(C_{i,B})$  consisting of the  $\text{Spec}(g_{i,B,j,B \cap \{j, \dots, n\}})$  for  $j \in A \setminus \{i\}$ . For such  $t$  set  $C_t := C_{i,B}$ . Clearly for  $j \in A$  we have a pull back functor  $z_{F_t}^r(C_t) \rightarrow z^r(C_{j,B \cap \{j, \dots, n\}})$  which for  $B' \subset \{j, \dots, n\}$  with  $B \cap \{j, \dots, n\} \subset B'$  we can prolong via smooth pull back to a map to  $z^r(C_{j,B'})$ .

**Lemma 5.9:** *Let  $t \rightarrow s$  be a map in  $\mathcal{E}_n$ . Then the above map  $z_{F_t}^r(C_t) \rightarrow z^r(C_s)$  factors through  $z_{F_s}^r(C_s)$ .*

*Proof.* Let  $t = (A, B, i)$  and  $s = (A', B', j)$ . Set  $s' := (A \cap \{j, \dots, n\}, B \cap \{j, \dots, n\}, j)$ . Without loss of generality we can assume  $A' = A \cap \{j, \dots, n\}$ . Clearly the map  $z_{F_t}^r(C_t) \rightarrow$

$z^r(C_{s'})$  factors through  $z_{F_{s'}}^r(C_{s'})$ . If  $A' = \{j\}$  we are done, otherwise fix  $k \in A' \setminus \{j\}$ . Set  $B'' := (B \cap \{j, \dots, n\}) \cup (B' \cap \{j, \dots, k\})$  and  $s'' := (A', B'', j)$ . Then we have a well defined map  $z_{F_{s'}}^r(C_{s'}) \rightarrow z_{\{g_{j,B'',k,B'' \cap \{k, \dots, n\}}\}}^r(C_{s''})$  since cycles meet in the correct codimension. Furthermore we have a well defined map  $z_{\{g_{j,B'',k,B'' \cap \{k, \dots, n\}}\}}^r(C_{s''}) \rightarrow z_{\{g_{j,B',k,B' \cap \{k, \dots, n\}}\}}^r(C_s)$  for the same reason. Altogether we see that cycles meet as claimed.  $\square$

For  $f: t \rightarrow s$  a map in  $\mathcal{E}_n$  we let  $\alpha_K(f): z_{F_t}^r(C_t) \rightarrow z_{F_s}^r(C_s)$  be the map constructed above using Lemma 5.9.

**Lemma 5.10:** *For  $f: t \rightarrow s$  and  $g: s \rightarrow r$  two maps in  $\mathcal{E}_n$  we have  $\alpha_K(g \circ f) = \alpha_K(g) \circ \alpha_K(f)$ .*

*Proof.* Let  $t = (A, B, i)$ ,  $s = (A', B', j)$  and  $r = (A'', B'', k)$ . Then the map  $\alpha_K(f)$  is defined by pulling back cycles via the map  $\text{Spec}(g_{i,B,j,B'})$  and the map  $\alpha_K(g)$  by pull back via  $\text{Spec}(g_{j,B',k,B''})$ . Thus the claim follows from (14).  $\square$

Setting  $\alpha_K(t) := z_{F_t}^r(C_t)$  for  $t$  an object of  $\mathcal{E}_n$  and using Lemma 5.10 we get a functor  $\alpha_K: \mathcal{E}_n \rightarrow \text{Cpx}(\text{Ab})$ .

By restricting everything to opens  $U$  in  $S_{Zar}$  we get a functor

$$\tilde{\alpha}_K: \mathcal{E}_n \rightarrow \text{Cpx}(\text{Sh}(S_{Zar}, \mathbb{Z})).$$

**Lemma 5.11:** *The functor  $\tilde{\alpha}_K$  sends weak equivalences in  $\mathcal{E}_n$  to quasi isomorphisms.*

*Proof.* This follows from Theorem 5.8, Theorem 3.14 and the fact that for  $X \in \text{Sm}_S$  the push forward of  $(X_{Zar} \ni Y \mapsto z^r(Y))$  via the structure morphism  $X \rightarrow S$  computes the derived push forward, which follows from [12, Theorem 1.7].  $\square$

**Lemma 5.12:** *Let  $f: [m] \rightarrow [n]$  be a monomorphism in  $\Delta$  and  $K$  a  $n$ -simplex in the nerve of  $\mathcal{S}$ . Then the composition  $\mathcal{E}_m \xrightarrow{f_*} \mathcal{E}_n \xrightarrow{\tilde{\alpha}_K} \text{Cpx}(\text{Sh}(S_{Zar}, \mathbb{Z}))$  is equal to  $\tilde{\alpha}_{f^*K}$ .*

*Proof.* We use a superscript  $K$  or  $f^*K$  to distinguish between the objects which are defined above for  $K$  respectively  $f^*K$ . We have  $C_t^{f^*K} = C_{f_*t}^K$  and  $F_t^{f^*K} = F_{f_*t}^K$  for  $t$  an object of  $\mathcal{E}_m$ . Thus the claim follows on objects. The definitions of the two functors on morphisms also coincide, thus the claim follows.  $\square$

For a category  $I$  we let  $\hat{I}$  be the subcategory of  $I \times \mathbb{N}$  (where  $\mathbb{N}$  is a category in the usual way) which has all objects and where a map  $(A, n) \rightarrow (B, m)$  belongs to  $\hat{I}$

if and only if the map  $A \rightarrow B$  is the identity or if  $m > n$ . Note that a composition of non-identity maps is again a non-identity map in  $\hat{I}$ .

We let a map  $(A, n) \rightarrow (B, m)$  in  $\hat{I}$  be a weak equivalence if and only if the map  $A \rightarrow B$  is the identity. We have a canonical projection functor  $p: \hat{I} \rightarrow I$ .

**Proposition 5.13:** *For any category  $I$  the canonical functor  $L^H \hat{I} \rightarrow I$  is a weak equivalence of simplicial categories.*

*Proof.* We use Lemma 5.7. Let  $\mathcal{C}$  be a model category and let  $\mathcal{C}^{\hat{I}}$  be equipped with the projective model structure (which exists since  $\hat{I}$  has the structure of a direct category). Let  $D: \hat{I} \rightarrow \mathcal{C}$  be a cofibrant diagram which preserves weak equivalences. For  $i \in I$  the diagram  $D|_{p/i}$  is also cofibrant by [18, Lemma 4.2] (it is not used here that  $\mathcal{C}^I$  also should have a model structure). The full subcategory  $J$  comprised by the  $((i, n), p((i, n)) \xrightarrow{\text{id}} i)$  in  $p/i$  is homotopy right cofinal, thus  $\text{colim}(D|_{p/i}) \simeq \text{hocolim}(D|_J)$  from which it follows that  $D \rightarrow r(l(D))$ , where  $l$  is the left adjoint to  $r: \mathcal{C}^I \rightarrow \mathcal{C}^{\hat{I}}$ , is a weak equivalence.  $\square$

Let  $\mathcal{N}$  be the nerve of  $\hat{\mathcal{S}}$  and  $\pi$  the nerve of the map  $p$  from above. For any  $K \in \mathcal{N}_n$  we let  $f_K: [n] \rightarrow [n']$  be the unique epimorphism in  $\Delta$  such that  $K = f_K^*(K')$  with  $K' \in \mathcal{N}_{n'}$  non-degenerate.  $K'$  is then also uniquely determined. We let  $\beta_K$  be the composition

$$\mathcal{E}_n \xrightarrow{f_{K,*}} \mathcal{E}_{n'} \xrightarrow{\tilde{\alpha}_{\pi(K')}} \text{Cpx}(\text{Sh}(S_{\text{Zar}}, \mathbb{Z})).$$

The reason for introducing  $\hat{\mathcal{S}}$  is the following observation.

**Lemma 5.14:** *Let  $h: [m] \rightarrow [n]$  be a map in  $\Delta$  and  $K \in \mathcal{N}_n$ . Then the composition  $\mathcal{E}_m \xrightarrow{h_*} \mathcal{E}_n \xrightarrow{\beta_K} \text{Cpx}(\text{Sh}(S_{\text{Zar}}, \mathbb{Z}))$  is equal to  $\beta_{h^*K}$ .*

*Proof.* Since every composition of non-identity maps in  $\hat{\mathcal{S}}$  is a non-identity map we have a commutative diagram

$$\begin{array}{ccc} [m] & \xrightarrow{h} & [n] \\ \downarrow f_{h^*K} & & \downarrow f_K \\ [m'] & \longrightarrow & [n'] \end{array}$$

where the bottom horizontal map is a monomorphism. Thus the claim follows from Lemma 5.12 and the definition of the maps  $\beta_K$  and  $\beta_{h^*K}$ .  $\square$

Let  $\Gamma: \text{Cpx}(\text{Sh}(S_{\text{Zar}}, \mathbb{Z})) \rightarrow \text{Cpx}(\mathbf{Ab})$  be a fibrant replacement functor followed by the global sections functor. We denote by  $qi$  the subcategory of quasi isomorphisms of

$\text{Cpx}(\text{Ab})$ . By Lemma 5.11 we get for any  $K \in \mathcal{N}_n$  induced functors  $L^H(\Gamma \circ \beta_K): L^H\mathcal{E}_n \rightarrow L_{qi}^H\text{Cpx}(\text{Ab})$  which are compatible with maps in  $\Delta$  by Lemma 5.14.

Let  $q_\bullet: Q_\bullet \rightarrow L^H\mathcal{E}$  be a map between cosimplicial objects in  $\mathbf{sCat}$ . For  $K \in \mathcal{N}_n$  let  $\gamma_K := L^H(\Gamma \circ \beta_K) \circ q_n$ . Then the  $\gamma_K$  are again compatible with maps in  $\Delta$ . By a coend construction we can pair a simplicial set  $L$  and any cosimplicial object  $P_\bullet$  in  $\mathbf{sCat}$  to obtain an object of  $\mathbf{sCat}$  which we denote by  $\mathcal{D}_{P_\bullet}^L$ . In the case  $L = \mathcal{N}$  we just write  $\mathcal{D}_{P_\bullet}$ . If  $L$  is the nerve of a category  $C$  we let  $\mathcal{D}_{P_\bullet}^C := \mathcal{D}_{P_\bullet}^L$ . We have  $\mathcal{D}_{[\bullet]} \cong \hat{\mathcal{S}}$ , thus we have a natural map  $\mathcal{D}_{Q_\bullet} \rightarrow \hat{\mathcal{S}}$ .

**Lemma 5.15:** *If  $Q_\bullet$  is Reedy cofibrant and the map  $Q_\bullet \rightarrow [\bullet]$  is a weak equivalence then the natural map  $\mathcal{D}_{Q_\bullet} \rightarrow \hat{\mathcal{S}}$  is a weak equivalence of simplicial categories.*

*Proof.* One deduces the result from the analogous statement for the usual adjunction between simplicial sets and simplicial categories involving the simplicial nerve functor, [15, Theorem 2.2.0.1].  $\square$

From now on suppose that  $Q_\bullet$  is Reedy cofibrant and that the map  $Q_\bullet \rightarrow [\bullet]$  is a weak equivalence (which can always be achieved by a cofibrant replacement of  $L^H\mathcal{E}$  in  $\mathbf{sCat}^\Delta$ ). The compatible maps  $\gamma_K$  give rise to an induced map  $\gamma: \mathcal{D}_{Q_\bullet} \rightarrow L_{qi}^H\text{Cpx}(\text{Ab})$ .

**Lemma 5.16:** *The map  $\gamma$  gives rise to a diagram  $\gamma' \in \text{Ho}(\text{Cpx}(\text{Ab})^{\hat{\mathcal{S}}})$  which is well-defined up to canonical isomorphism.*

*Proof.* This follows from a strictification result, see [15, Proposition 4.2.4.4].  $\square$

We define the motivic complex  $\mathcal{M}(r)$  to be the push forward of  $\gamma'[-2r]$  with respect to the composition  $\text{Ho}(\text{Cpx}(\text{Ab})^{\hat{\mathcal{S}}}) \rightarrow \text{Ho}(\text{Cpx}(\text{Ab})^{\mathcal{S}}) \rightarrow \text{D}(\text{Sh}(\text{Sm}_{S, \text{Zar}}, \mathbb{Z}))$ , where the first map is induced by  $p: \hat{\mathcal{S}} \rightarrow \mathcal{S}$  and the second map is the Zariski localization map.

## 5.2 Properties of the motivic complexes

Let  $\mathcal{C}, \mathcal{S}'$  be categories and  $I$  a small category. Let  $\mathcal{E}'$  be a cosimplicial object in  $\mathbf{sCat}$  over  $[\bullet]$ . Let for any  $n$ -simplex  $K$  of the nerve of  $\mathcal{S}'$  be a functor  $\alpha_K: \mathcal{E}'_n \times I \rightarrow \mathcal{C}$  be given. Suppose these functors are compatible for monomorphisms in  $\Delta$ , i.e. that for  $f: [m] \rightarrow [n]$  a monomorphism we have  $\alpha_K \circ (f_* \times \text{id}) = \alpha_{f^*K}$ . Then for  $\tilde{K}$  a  $n$ -simplex of the nerve of  $\mathcal{S}' \times I$  we let  $T(\alpha)_{\tilde{K}}$  be the composition  $\mathcal{E}'_n \rightarrow \mathcal{E}'_n \times I \xrightarrow{\alpha_K} \mathcal{C}$ , where the second component of the first map is the composition  $\mathcal{E}'_n \rightarrow [n] \rightarrow I$  (the second map being the second component of  $\tilde{K}$ ) and where  $K$  is the first component of  $\tilde{K}$ . The  $T(\alpha)_{\tilde{K}}$  are then again compatible for monomorphisms in  $\Delta$ .

Let  $p: \mathcal{S}'' \rightarrow \mathcal{S}' \times I$  be a functor and suppose that the composition in  $\mathcal{S}''$  of two non-identity maps is a non-identity map. Let  $K$  be a  $n$ -simplex of the nerve of  $\mathcal{S}''$ . Let  $f: [n] \rightarrow [n']$  be the unique epimorphism in  $\Delta$  such that  $K = f^*(K')$  for a non-degenerate  $n'$ -simplex  $K'$ . Let  $T^p(\alpha)_K$  be the composition  $\mathcal{E}'_n \xrightarrow{f_*} \mathcal{E}'_{n'} \xrightarrow{T(\alpha)_{\tilde{K}}} \mathcal{C}$ , where  $\tilde{K}$  is the image of  $K'$  in the nerve of  $\mathcal{S}' \times I$ . Then the  $T^p(\alpha)_K$  are compatible for all maps in  $\Delta$ .

In our applications  $\mathcal{S}''$  will be  $\hat{\mathcal{S}}' \times I$ .

### 5.2.1 Comparison to flat maps

Let the notation be as in the last section. We denote by  $\mathcal{S}^{\text{fl}}$  the subcategory of  $\mathcal{S}$  consisting of flat maps.

Let  $K$  be a  $n$ -simplex in the nerve of  $\mathcal{S}^{\text{fl}}$ . In particular we have a chain  $A_0 \rightarrow \dots \rightarrow A_n$  of smooth  $D$ -algebras where each map is flat. We associate to this the functor  $\alpha'_K: [n] \rightarrow \text{Cpx}(\text{Sh}(S_{Zar}, \mathbb{Z}))$  which sends  $i$  to  $(U \mapsto z^r(\text{Spec}(A_i) \times_S U))$  and where the maps are induced by flat pullback of cycles. We denote by  $\tilde{\alpha}'_K$  the composition  $\mathcal{E}_n \xrightarrow{\varphi_n} [n] \xrightarrow{\alpha'_K} \text{Cpx}(\text{Sh}(S_{Zar}, \mathbb{Z}))$ .

Recall the maps  $\tilde{\alpha}_K$ . We have a natural transformation  $\tilde{\alpha}'_K \rightarrow \tilde{\alpha}_K$  which is induced by the maps  $A_i \rightarrow C_t$  for  $t = (A, B, i) \in \mathcal{E}_n$ . We note that the cycle conditions given by the  $F_t$  are fulfilled since for a map  $t \rightarrow s$  in  $\mathcal{E}_n$  with  $s = (A', B', j)$  the diagram

$$\begin{array}{ccc} C_t & \longrightarrow & C_s \\ \uparrow & & \uparrow \\ A_i & \longrightarrow & A_j \end{array}$$

commutes.

We denote by  $\alpha_K: \mathcal{E}_n \times [1] \rightarrow \text{Cpx}(\text{Sh}(S_{Zar}, \mathbb{Z}))$  the functor corresponding to this natural transformation.

Thus as in the beginning of section 5.2 we get a compatible family of maps  $T^p(\alpha)_K$ , where  $p$  is the functor  $\hat{\mathcal{S}}^{\text{fl}} \times [1] \rightarrow \mathcal{S}^{\text{fl}} \times [1]$ .

For  $K$  a  $n$ -simplex in the nerve of  $\hat{\mathcal{S}}^{\text{fl}} \times [1]$  let  $\tilde{\gamma}_K := L^H(\Gamma \circ T^p(\alpha)_K) \circ q_n$ .

The  $\tilde{\gamma}_K$  glue to give a map

$$\tilde{\gamma}: \mathcal{D}_{Q_\bullet}^{\hat{\mathcal{S}}^{\text{fl}} \times [1]} \rightarrow L_{qi}^H \text{Cpx}(\text{Ab}).$$

We denote by  $\tilde{\gamma}' \in \text{Ho}(\text{Cpx}(\text{Ab})^{\hat{\mathcal{S}}^{\text{fl}} \times [1]})$  the diagram canonically associated to  $\tilde{\gamma}$ .

**Lemma 5.17:**  $\gamma'|_{\hat{\mathcal{S}}^{\text{fl}}}$  and  $\tilde{\gamma}'|_{\hat{\mathcal{S}}^{\text{fl}} \times \{1\}}$  are canonically isomorphic.

*Proof.* This follows by construction of  $\gamma'$  and  $\tilde{\gamma}'$ .  $\square$

**Lemma 5.18:**  $\tilde{\gamma}'|_{\hat{\mathcal{S}}^{\text{fl}} \times \{0\}}$  is canonically isomorphic to the diagram on  $\hat{\mathcal{S}}^{\text{fl}}$  which associates to an  $(A, n, (a_1, \dots, a_n), m)$  the cycle complex  $z^r(A)$ .

*Proof.* This follows by construction of  $\tilde{\gamma}'$ .  $\square$

Let  $\text{Sm}_S^{\text{fl}}$  be the subcategory of  $\text{Sm}_S$  of flat maps.

**Corollary 5.19:** The complex  $\mathcal{M}(r)|_{\text{Sm}_S^{\text{fl}}}$  is canonically isomorphic to the diagram  $X \mapsto z^r(X)[-2r]$  in  $\text{D}(\text{Sh}(\text{Sm}_{S, \text{Zar}}^{\text{fl}}, \mathbb{Z}))$ .

*Proof.* This follows from Lemmas 5.17 and 5.18, the fact that the map in  $\text{Ho}(\text{Cpx}(\text{Ab})^{\hat{\mathcal{S}}^{\text{fl}}})$  associated to  $\tilde{\gamma}'$  is an isomorphism, the fact (which follows from these Lemmas) that the push forward of  $\gamma'$  with respect to  $\text{Ho}(\text{Cpx}(\text{Ab})^{\hat{\mathcal{S}}}) \rightarrow \text{Ho}(\text{Cpx}(\text{Ab})^{\mathcal{S}})$  has Zariski descent and from Proposition 5.13 (or better its proof).  $\square$

**Corollary 5.20:** For  $X \in \text{Sm}_S$  there is a canonical isomorphism  $\mathcal{M}^X(r) \cong \mathcal{M}(r)|_{X_{\text{Zar}}}$  in  $\text{D}(\text{Sh}(X_{\text{Zar}}, \mathbb{Z}))$ .

*Proof.* This follows from Corollary 5.19.  $\square$

### 5.2.2 Some localization triangles

We still keep the notation of section 5.1. Let  $A$  be a smooth  $D$ -algebra. Let  $K$  be a  $n$ -simplex in the nerve of  $\mathcal{S}$ . For  $t \in \mathcal{E}_n$  set  $C_t^A := A \otimes_D C_t$  and  $F_t^A := \{\text{Spec}(A) \times_S a | a \in F_t\}$ . Then as in section 5.1 we get functors  $\alpha_K^A: \mathcal{E}_n \rightarrow \text{Cpx}(\text{Ab})$ ,  $t \mapsto z_{F_t^A}^r(C_t^A)$ , and  $\tilde{\alpha}_K^A: \mathcal{E}_n \rightarrow \text{Cpx}(\text{Sh}(S_{\text{Zar}}, \mathbb{Z}))$ .

Now let  $a_1, \dots, a_k \in A$  be generators of  $A$ . Set  $\underline{A} := (A, k, (a_1, \dots, a_k)) \in \mathcal{S}$ . For  $(A', k', (a'_1, \dots, a'_{k'})) \in \mathcal{S}$  let  $\underline{A} \otimes (A', k', (a'_1, \dots, a'_{k'})) := (A \otimes_D A', k + k', (a_1 \otimes 1, \dots, a_k \otimes 1, 1 \otimes a'_1, \dots, 1 \otimes a'_{k'})) \in \mathcal{S}$ . Similarly for a  $n$ -simplex  $K$  in the nerve of  $\mathcal{S}$  the  $n$ -simplex  $\underline{A} \otimes K$  is defined.

For  $K$  a  $n$ -simplex in the nerve of  $\mathcal{S}$  we have a natural transformation  $\tilde{\alpha}_K^A \rightarrow \tilde{\alpha}_{\underline{A} \otimes K}^A$  induced by the obvious inclusion maps of algebras. We denote by  $\bar{\alpha}_K^A: \mathcal{E}_n \times [1] \rightarrow \text{Cpx}(\text{Sh}(S_{\text{Zar}}, \mathbb{Z}))$  the functor corresponding to this natural transformation.

For  $K$  a  $n$ -simplex in the nerve of  $\hat{\mathcal{S}} \times [1]$  we let  $\gamma_K^A := L^H(\Gamma \circ T^p(\bar{\alpha}^A)_K) \circ q_n$ , where  $p$  is the functor  $\hat{\mathcal{S}} \times [1] \rightarrow \mathcal{S} \times [1]$ .

The  $\gamma_K^A$  glue to give a map

$$\gamma^A: \mathcal{D}_{Q_\bullet}^{\hat{\mathcal{S}} \times [1]} \rightarrow L_{qi}^H \text{Cpx}(\text{Ab}).$$

We denote by  $\gamma'_A \in \text{Ho}(\text{Cpx}(\text{Ab})^{\hat{\mathcal{S}} \times [1]})$  the diagram canonically associated to  $\gamma^A$ .

**Lemma 5.21:** *The push forward of  $\gamma'_A[-2r]|_{\hat{\mathcal{S}} \times \{1\}}$  to  $\text{D}(\text{Sh}(\text{Sm}_{S, \text{Zar}}, \mathbb{Z}))$  is canonically isomorphic to  $\underline{\mathbb{R}\text{Hom}}(\mathbb{Z}[\text{Spec}(A)]_{\text{Zar}}, \mathcal{M}(r))$ .*

*Proof.* This follows from the definition of  $\gamma'_A$ .  $\square$

**Corollary 5.22:** *The push forward of  $\gamma'_A[-2r]|_{\hat{\mathcal{S}} \times \{0\}}$  to  $\text{D}(\text{Sh}(\text{Sm}_{S, \text{Zar}}, \mathbb{Z}))$  is canonically isomorphic to  $\underline{\mathbb{R}\text{Hom}}(\mathbb{Z}[\text{Spec}(A)]_{\text{Zar}}, \mathcal{M}(r))$ .*

*Proof.* This follows from the fact that the map in  $\text{Ho}(\text{Cpx}(\text{Ab})^{\hat{\mathcal{S}}})$  associated to  $\gamma'_A$  is an isomorphism.  $\square$

Now let  $f: A \rightarrow A'$  be a flat map to a smooth  $D$ -algebra  $A'$ , let  $a'_1, \dots, a'_{k'} \in A'$  be generators. We have functors  $a: \mathcal{S} \rightarrow \mathcal{S}$ ,  $(B, l, (b_1, \dots, b_l)) \mapsto (A \otimes_D B, k + l, (a_1 \otimes 1, \dots, a_k \otimes 1, 1 \otimes b_1, \dots, 1 \otimes b_l))$  and  $b: \mathcal{S} \rightarrow \mathcal{S}$ ,  $(B, l, (b_1, \dots, b_l)) \mapsto (A' \otimes_D B, k' + l, (a'_1 \otimes 1, \dots, a'_{k'} \otimes 1, 1 \otimes b_1, \dots, 1 \otimes b_l))$  and a natural transformation  $a \rightarrow b$  induced by  $f$ . We let  $G: \mathcal{S} \times [1] \rightarrow \mathcal{S}$  be the corresponding functor.

Let  $K$  be a  $n$ -simplex in the nerve of  $\mathcal{S} \times [1]$ . Let  $\alpha_{2,K}^f := \tilde{\alpha}_{G(K)}^f$ .

We have a natural transformation  $\tilde{\alpha}_K^A \rightarrow \tilde{\alpha}_K^{A'}$  induced by  $f$ . Let  $\alpha_{1,K}^f: \mathcal{E}_n \times [1] \rightarrow \text{Cpx}(\text{Sh}(S_{\text{Zar}}, \mathbb{Z}))$  be the corresponding functor.

We have a natural transformation  $\alpha_{1,K}^f \rightarrow \alpha_{2,K}^f$  induced by the natural inclusion maps of algebras. We denote by  $\bar{\alpha}_K^f: \mathcal{E}_n \times [1]^2 \rightarrow \text{Cpx}(\text{Sh}(S_{\text{Zar}}, \mathbb{Z}))$  the corresponding functor.

For  $K$  a  $n$ -simplex in the nerve of  $\hat{\mathcal{S}} \times [1]^2$  we let  $\gamma_K^f := L^H(\Gamma \circ T^p(\bar{\alpha}^f)_K) \circ q_n$ , where  $p$  is the functor  $\hat{\mathcal{S}} \times [1]^2 \rightarrow \mathcal{S} \times [1]^2$ .

The  $\gamma_K^f$  glue to give a map

$$\gamma^f: \mathcal{D}_{Q_\bullet}^{\hat{\mathcal{S}} \times [1]^2} \rightarrow L_{qi}^H \text{Cpx}(\text{Ab}).$$

We denote by  $\gamma'_f \in \text{Ho}(\text{Cpx}(\text{Ab})^{\hat{\mathcal{S}} \times [1]^2})$  the diagram canonically associated to  $\gamma^f$ .

**Lemma 5.23:** *The push forward to  $\text{D}(\text{Sh}(\text{Sm}_{S, \text{Zar}}, \mathbb{Z}))$  of the map in  $\text{Ho}(\text{Cpx}(\text{Ab})^{\hat{\mathcal{S}}})$  associated to  $\gamma'_f[-2r]|_{\hat{\mathcal{S}} \times [1] \times \{1\}}$  is canonically isomorphic to the map*

$$\underline{\mathbb{R}\text{Hom}}(\mathbb{Z}[\text{Spec}(A)]_{\text{Zar}}, \mathcal{M}(r)) \rightarrow \underline{\mathbb{R}\text{Hom}}(\mathbb{Z}[\text{Spec}(A')]_{\text{Zar}}, \mathcal{M}(r)).$$



*Proof.* This follows from the definition of  $\gamma'_f$ .  $\square$

**Corollary 5.24:** *The push forward to  $D(\mathrm{Sh}(\mathrm{Sm}_{S,Zar}, \mathbb{Z}))$  of the map in  $\mathrm{Ho}(\mathrm{Cpx}(\mathrm{Ab})^{\hat{S}})$  associated to  $\gamma'_f[-2r]|_{\hat{S} \times [1] \times \{0\}}$  is canonically isomorphic to the map*

$$\underline{\mathrm{RHom}}(\mathbb{Z}[\mathrm{Spec}(A)]_{Zar}, \mathcal{M}(r)) \rightarrow \underline{\mathrm{RHom}}(\mathbb{Z}[\mathrm{Spec}(A')]_{Zar}, \mathcal{M}(r)).$$

**Proposition 5.25:** *Let  $i: Z \rightarrow X$  be a closed immersion of affine schemes in  $\mathrm{Sm}_S$  of codimension 1 with open affine complement  $U$ . Then there is an exact triangle*

$$\begin{aligned} \underline{\mathrm{RHom}}(\mathbb{Z}[Z]_{Zar}, \mathcal{M}(r-1))[-2] &\rightarrow \underline{\mathrm{RHom}}(\mathbb{Z}[X]_{Zar}, \mathcal{M}(r)) \\ \rightarrow \underline{\mathrm{RHom}}(\mathbb{Z}[U]_{Zar}, \mathcal{M}(r)) &\rightarrow \underline{\mathrm{RHom}}(\mathbb{Z}[Z]_{Zar}, \mathcal{M}(r-1))[-1] \end{aligned}$$

in  $D(\mathrm{Sh}(\mathrm{Sm}_{S,Zar}, \mathbb{Z}))$ , where the second map is induced by the morphism  $U \rightarrow X$ .

*Proof.* Let  $A \rightarrow A''$  be the map of function algebras corresponding to  $i$  and  $A \rightarrow A'$  the map corresponding to the open inclusion  $U \subset X$ .

For  $K$  a  $n$ -simplex in the nerve of  $\mathcal{S}$  we define a functor  $\alpha_K^\circ: \mathcal{E}_n \times [1]^2 \rightarrow \mathrm{Cpx}(\mathrm{Ab})$  by sending  $(t, 0, 0)$  to  $z_{F_t^{A''}}^{r-1}(C_t^{A''})$ ,  $(t, 1, 0)$  to  $z_{F_t^A}^r(C_t^A)$ ,  $(t, 1, 1)$  to  $z_{F_t^{A'}}^r(C_t^{A'})$  and  $(t, 0, 1)$  to 0. Sheafification on  $S$  yields a functor  $\tilde{\alpha}_K^\circ: \mathcal{E}_n \times [1]^2 \rightarrow \mathrm{Cpx}(\mathrm{Sh}(S_{Zar}, \mathbb{Z}))$ .

For  $K$  a  $n$ -simplex in the nerve of  $\hat{\mathcal{S}} \times [1]^2$  let  $\gamma_K^\circ := L^H(\Gamma \circ T^p(\tilde{\alpha}^\circ)_K) \circ q_n$ .

The  $\gamma_K^\circ$  glue to give a map

$$\gamma^\circ: \mathcal{D}_{Q_\bullet}^{\hat{S} \times [1]^2} \rightarrow L_{qi}^H \mathrm{Cpx}(\mathrm{Ab}).$$

We denote by  $\gamma'_\alpha \in \mathrm{Ho}(\mathrm{Cpx}(\mathrm{Ab})^{\hat{S} \times [1]^2})$  the diagram canonically associated to  $\gamma^\circ$ .

The square in  $D(\mathrm{Sh}(\mathrm{Sm}_{S,Zar}, \mathbb{Z}))$  associated to the push forward of  $\gamma'_\alpha[-2r]$  is exact by [12, Theorem 1.7]. Moreover by Corollary 5.22 the entries in this square in the places  $(0, 0)$ ,  $(1, 0)$  and  $(1, 1)$  are  $\underline{\mathrm{RHom}}(\mathbb{Z}[Z]_{Zar}, \mathcal{M}(r-1))[-2]$ ,  $\underline{\mathrm{RHom}}(\mathbb{Z}[X]_{Zar}, \mathcal{M}(r))$  and  $\underline{\mathrm{RHom}}(\mathbb{Z}[U]_{Zar}, \mathcal{M}(r))$ , and the map from entry  $(1, 0)$  to  $(1, 1)$  is the one induced by the map  $U \subset X$  by Corollary 5.24. Thus by [14, Definition 1.1.2.11] we get the exact triangle as required.  $\square$

### 5.2.3 The étale cycle class map

For  $X \in \mathrm{Sm}_S$  and  $F$  a finite set of closed immersions in  $\mathrm{Sm}_S$  with target  $X$  we denote by  $c_F^r(X, n)$  the set of cycles (closed integral subschemes) of  $X \times \Delta^n$  which intersect all  $Z \times Y$  with  $Z \in F \cup \{X\}$  and  $Y$  a face of  $\Delta^n$  properly.

Let  $U := S[\frac{1}{p}]$ . Let  $m$  be an integer prime to  $p$ . Let  $\mu_m^{\otimes r} \rightarrow \mathcal{G}$  be an injectively fibrant replacement in  $\text{Cpx}(\text{Sh}(\text{Sm}_U, \text{ét}, \mathbb{Z}/m))$ .

Let  $X \in \text{Sm}_U$ . For  $W$  a closed subset of  $X$  such that each irreducible component has codimension greater or equal to  $r$  set  $\mathcal{G}^W(X) := \ker(\mathcal{G}(X) \rightarrow \mathcal{G}(X \setminus W))$ .

As in [11, 12.3] there is a canonical isomorphism of  $H^{2r}(\mathcal{G}^W(X))$  with the free  $\mathbb{Z}/m$ -module on the irreducible components of  $W$  of codimension  $r$  and the map  $\tau_{\leq 2r} \mathcal{G}^W(X) \rightarrow H^{2r}(\mathcal{G}^W(X))[-2r]$  is a quasi isomorphism.

For  $F$  a finite set of closed immersions in  $\text{Sm}_U$  with target  $X$  denote by  $\mathcal{G}_F^r(X, n)$  the colimit of the  $\mathcal{G}^W(X \times \Delta^n)$  where  $W$  runs through the finite unions of elements of  $c_F^r(X, n)$ . The simplicial complex of  $\mathbb{Z}/m$ -modules  $\tau_{\leq 2r} \mathcal{G}_F^r(X, \bullet)$  augments to the simplicial abelian group  $z_F^r(X, \bullet)/m[-2r]$ . This augmentation is a levelwise quasi isomorphism. We denote by  $\mathcal{G}_F^r(X)$  the total complex associated to the double complex which is the normalized complex associated to  $\tau_{\leq 2r} \mathcal{G}_F^r(X, \bullet)$ . Thus we get a quasi isomorphism  $\mathcal{G}_F^r(X) \rightarrow z_F^r(X)/m[-2r]$ .

On the other hand we have a canonical map  $\mathcal{G}_F^r(X, n) \rightarrow \mathcal{G}(X \times \Delta^n)$  compatible with the simplicial structure. We denote by  $\mathcal{G}'(X)$  the total complex associated to the double complex which is the normalized complex associated to  $\mathcal{G}(X \times \Delta^\bullet)$ . We have a canonical quasi isomorphism  $\mathcal{G}(X) \rightarrow \mathcal{G}'(X)$  and a canonical map  $\mathcal{G}_F^r(X) \rightarrow \mathcal{G}'(X)$ .

Thus in  $\text{D}(\mathbb{Z}/m)$  we get a map

$$z_F^r(X)/m[-2r] \cong \mathcal{G}_F^r(X) \rightarrow \mathcal{G}'(X) \cong \mathcal{G}(X).$$

Our next aim is to make this assignment functorial in  $X$  for all maps in  $\text{Sm}_U$ . In the following we sometimes insert into the above definitions  $A$  instead of  $\text{Spec}(A)$ . Let  $I$  be the category  $0 \leftarrow 1 \rightarrow 2 \leftarrow 3 \leftarrow 4$ . We denote by  $\mathcal{S}_p$  the full subcategory of  $\mathcal{S}$  such that  $p$  is invertible in the algebras belonging to the objects. We use the notation of section 5.1. Let  $K$  be a  $n$ -simplex in the nerve of  $\mathcal{S}_p$ . We assign to  $K$  the following functor  $\alpha'_K: \mathcal{E}_n \times I \rightarrow \text{Cpx}(\text{Ab})$ :  $(t, 0) \mapsto \alpha_K(t)/m[-2r]$ ,  $(t, 1) \mapsto \mathcal{G}_{F_t}^r(C_t)$ ,  $(t, 2) \mapsto \mathcal{G}'(C_t)$ ,  $(t, 3) \mapsto \mathcal{G}(C_t)$ ,  $(t, 4) \mapsto \mathcal{G}(A_{\varphi_n(t)})$ . Sheafifying on  $U_{Zar}$  yields a functor  $\tilde{\alpha}'_K: \mathcal{E}_n \times I \rightarrow \text{Cpx}(\text{Sh}(U_{Zar}, \mathbb{Z}))$ . These functors are compatible for monomorphisms in  $\Delta$ .

For  $K$  a  $n$ -simplex of the nerve of  $\hat{\mathcal{S}}_p \times I$  let  $\gamma_K^I := L^H(\Gamma \circ T^p(\tilde{\alpha}')_K) \circ q_n$ , where  $p$  is the functor  $\hat{\mathcal{S}}_p \times I \rightarrow \mathcal{S}_p \times I$ . The  $\gamma_K^I$  glue to give a map

$$\gamma^I: \mathcal{D}_{Q_\bullet}^{\hat{\mathcal{S}}_p \times I} \rightarrow L_{qi}^H \text{Cpx}(\text{Ab}).$$

We denote by  $\gamma'_I \in \text{Ho}(\text{Cpx}(\text{Ab})^{\hat{\mathcal{S}}_p \times I})$  the diagram canonically associated to  $\gamma^I$ .

The push forward of the  $I$ -diagram in  $\mathrm{Ho}(\mathrm{Cpx}(\mathrm{Ab})^{\hat{\mathcal{S}}_p})$  corresponding to the diagram  $\gamma'_I$  to  $\mathrm{D}(\mathrm{Sh}(\mathrm{Sm}_{U,Zar}, \mathbb{Z}))$  is an  $I$ -diagram of the form  $(\mathcal{M}(r)/m)|_U \cong \bullet \rightarrow \bullet \cong \bullet \cong \mathbb{R}\epsilon_* \mu_m^{\otimes r}$  which yields the cycle class map.

Next we wish to show the compatibility of this cycle class map with the original cycle class map defined for flat morphisms.

If in the following notation a collection  $F$  of closed subschemes is missing we assume that this  $F$  is empty. For  $K$  a  $n$ -simplex in the nerve of  $\mathcal{S}_p^{\mathrm{fl}}$  (with the obvious notation) we define a functor  $\alpha''_K: \mathcal{E}_n \times I \rightarrow \mathrm{Cpx}(\mathrm{Ab})$  in the following way:  $(t, 0) \mapsto z^r(A_{\varphi_n(t)})[-2r]$ ,  $(t, 1) \mapsto \mathcal{G}^r(A_{\varphi_n(t)})$ ,  $(t, 2) \mapsto \mathcal{G}'(A_{\varphi_n(t)})$ ,  $(t, 3), (t, 4) \mapsto \mathcal{G}(A_{\varphi_n(t)})$ . Sheafifying on  $U$  yields a functor  $\tilde{\alpha}''_K: \mathcal{E}_n \times I \rightarrow \mathrm{Cpx}(\mathrm{Sh}(U_{Zar}, \mathbb{Z}))$ . There is an obvious natural transformation  $\tilde{\alpha}''_K \rightarrow \tilde{\alpha}'_K$ . We denote by  $\bar{\alpha}_K: \mathcal{E}_n \times I \times [1] \rightarrow \mathrm{Cpx}(\mathrm{Sh}(U_{Zar}, \mathbb{Z}))$  the corresponding functor.

For  $K$  a  $n$ -simplex of the nerve of  $\hat{\mathcal{S}}_p^{\mathrm{fl}} \times I \times [1]$  let  $\gamma_K^{I \times [1]} := L^H(\Gamma \circ T^p(\bar{\alpha})_K) \circ q_n$ , where  $p$  is the functor  $\hat{\mathcal{S}}_p^{\mathrm{fl}} \times I \times [1] \rightarrow \mathcal{S}_p^{\mathrm{fl}} \times I \times [1]$ . The  $\gamma_K^{I \times [1]}$  glue to give a map

$$\gamma^{I \times [1]}: \mathcal{D}_{Q_\bullet}^{\hat{\mathcal{S}}_p^{\mathrm{fl}} \times I \times [1]} \rightarrow L_{qi}^H \mathrm{Cpx}(\mathrm{Ab}).$$

We denote by  $\gamma'_{I \times [1]} \in \mathrm{Ho}(\mathrm{Cpx}(\mathrm{Ab})^{\hat{\mathcal{S}}_p^{\mathrm{fl}} \times I \times [1]})$  the diagram canonically associated to  $\gamma^{I \times [1]}$ .

The push forward of the  $I \times [1]$ -diagram in  $\mathrm{Ho}(\mathrm{Cpx}(\mathrm{Ab})^{\hat{\mathcal{S}}_p^{\mathrm{fl}}})$  corresponding to the diagram  $\gamma'_{I \times [1]}$  to  $\mathrm{D}(\mathrm{Sh}(\mathrm{Sm}_{U,Zar}^{\mathrm{fl}}, \mathbb{Z}))$  is an  $I \times [1]$ -diagram where the subdiagram indexed on  $I \times \{0\}$  gives the old cycle class map and the subdiagram indexed on  $I \times \{1\}$  the new cycle class map restricted to flat maps. Thus the two cycle class maps are canonically isomorphic (over flat maps).

**Corollary 5.26:** *For  $X \in \mathrm{Sm}_U$  the cycle class map  $\mathcal{M}^X(r)/m \rightarrow \mathbb{R}\epsilon_* \mathbb{Z}/m(r)$  from section 3 is canonically isomorphic to the cycle class map  $(\mathcal{M}(r)/m)|_U \rightarrow \mathbb{R}\epsilon_* \mu_m^{\otimes r}$  restricted to  $X_{Zar}$ .*

Now we also use the notation of section 5.2.2. We assume  $p$  is invertible in  $A$ . For a  $n$ -simplex  $K$  of the nerve of  $\mathcal{S}_p$  we define a functor  $(\alpha')_K^A: \mathcal{E}_n \times I \rightarrow \mathrm{Cpx}(\mathrm{Ab})$  in the following way:  $(t, 0) \mapsto \alpha_K^A(t)/m[-2r]$ ,  $(t, 1) \mapsto \mathcal{G}_{F_t^A}^r(C_t^A)$ ,  $(t, 2) \mapsto \mathcal{G}'(C_t^A)$ ,  $(t, 3) \mapsto \mathcal{G}(C_t^A)$ ,  $(t, 4) \mapsto \mathcal{G}(A \otimes_D A_{\varphi_n(t)})$ . Sheafifying on  $U_{Zar}$  yields a functor  $(\tilde{\alpha}')_K^A: \mathcal{E}_n \times I \rightarrow \mathrm{Cpx}(\mathrm{Sh}(U_{Zar}, \mathbb{Z}))$ . These functors are compatible for monomorphisms in  $\Delta$ .

We have a natural transformation  $(\tilde{\alpha}')_K^A \rightarrow \tilde{\alpha}'_{\underline{A} \otimes K}$  induced by the obvious inclusion maps of algebras. We denote by  $\bar{\alpha}_K^A: \mathcal{E}_n \times I \times [1] \rightarrow \mathrm{Cpx}(\mathrm{Sh}(U_{Zar}, \mathbb{Z}))$  the corresponding functor.

For  $K$  a  $n$ -simplex of the nerve of  $\hat{\mathcal{S}}_p \times I \times [1]$  let  $\gamma_K^{A, I \times [1]} := L^H(\Gamma \circ T^p(\bar{\alpha}^A)_K) \circ q_n$ , where  $p$  is the functor  $\hat{\mathcal{S}}_p \times I \times [1] \rightarrow \mathcal{S}_p \times I \times [1]$ . The  $\gamma_K^{A, I \times [1]}$  glue to give a map

$$\gamma^{A, I \times [1]}: \mathcal{D}_{Q_\bullet}^{\hat{\mathcal{S}}_p \times I \times [1]} \rightarrow L_{qi}^H \text{Cpx}(\text{Ab}).$$

We denote by  $\gamma'_{A, I \times [1]} \in \text{Ho}(\text{Cpx}(\text{Ab})^{\hat{\mathcal{S}}_p \times I \times [1]})$  the diagram canonically associated to  $\gamma^{A, I \times [1]}$ .

The push forward of the  $I \times [1]$ -diagram in  $\text{Ho}(\text{Cpx}(\text{Ab})^{\hat{\mathcal{S}}_p})$  corresponding to the diagram  $\gamma'_{A, I \times [1]}$  to  $\text{D}(\text{Sh}(\text{Sm}_{U, \text{Zar}}, \mathbb{Z}))$  is an  $I \times [1]$ -diagram where the subdiagram indexed on  $I \times \{1\}$  gives the functor  $\underline{\mathbb{R}\text{Hom}}(\text{Spec}(A), -)$  applied to the cycle class map  $(\mathcal{M}(r)/m)|_U \rightarrow \mathbb{R}\epsilon_* \mu_m^{\otimes r}$ .

**Corollary 5.27:** *The subdiagram indexed on  $I \times \{0\}$  of the above  $I \times [1]$  diagram in  $\text{D}(\text{Sh}(\text{Sm}_{U, \text{Zar}}, \mathbb{Z}))$  yields a map canonically isomorphic to the map*

$$\underline{\mathbb{R}\text{Hom}}(\text{Spec}(A), (\mathcal{M}(r)/m)|_U) \rightarrow \underline{\mathbb{R}\text{Hom}}(\text{Spec}(A), \mathbb{R}\epsilon_* \mu_m^{\otimes r})$$

*induced by the cycle class map.*

Let  $i: Z \rightarrow X$  be a closed immersion of affine schemes in  $\text{Sm}_S$  of codimension 1 with open affine complement  $V$ . The exact triangle

$$\begin{aligned} \underline{\mathbb{R}\text{Hom}}(\mathbb{Z}[Z]_{\text{Zar}}, \mathcal{M}(r-1))[-2] &\rightarrow \underline{\mathbb{R}\text{Hom}}(\mathbb{Z}[X]_{\text{Zar}}, \mathcal{M}(r)) \\ &\rightarrow \underline{\mathbb{R}\text{Hom}}(\mathbb{Z}[U]_{\text{Zar}}, \mathcal{M}(r)) \rightarrow \underline{\mathbb{R}\text{Hom}}(\mathbb{Z}[Z]_{\text{Zar}}, \mathcal{M}(r-1))[-1] \end{aligned}$$

in  $\text{D}(\text{Sh}(\text{Sm}_S, \mathbb{Z}))$  from Proposition 5.25 yields an exact triangle

$$\begin{aligned} \underline{\mathbb{R}\text{Hom}}(\mathbb{Z}[Z_U]_{\text{Zar}}, (\mathcal{M}(r-1)/m)|_U)[-2] &\rightarrow \underline{\mathbb{R}\text{Hom}}(\mathbb{Z}[X_U]_{\text{Zar}}, (\mathcal{M}(r)/m)|_U) \\ &\rightarrow \underline{\mathbb{R}\text{Hom}}(\mathbb{Z}[V_U]_{\text{Zar}}, (\mathcal{M}(r)/m)|_U) \rightarrow \underline{\mathbb{R}\text{Hom}}(\mathbb{Z}[Z_U]_{\text{Zar}}, (\mathcal{M}(r-1)/m)|_U)[-1] \end{aligned}$$

in  $\text{D}(\text{Sh}(\text{Sm}_{U, \text{Zar}}, \mathbb{Z}))$ .

**Proposition 5.28:** *Let the notation be as above. Then the diagram*

$$\begin{array}{ccc}
\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z_U]_{\mathrm{Zar}}, (\mathcal{M}(r-1)/m)|_U)[-2] & \longrightarrow & \mathbb{R}\epsilon_* \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z_U]_{\acute{e}t}, \mu_m^{\otimes(r-1)})[-2] \\
\downarrow & & \downarrow \\
\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X_U]_{\mathrm{Zar}}, (\mathcal{M}(r)/m)|_U) & \longrightarrow & \mathbb{R}\epsilon_* \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[X_U]_{\acute{e}t}, \mu_m^{\otimes r}) \\
\downarrow & & \downarrow \\
\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[V_U]_{\mathrm{Zar}}, (\mathcal{M}(r)/m)|_U) & \longrightarrow & \mathbb{R}\epsilon_* \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[V_U]_{\acute{e}t}, \mu_m^{\otimes r}) \\
\downarrow & & \downarrow \\
\underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z_U]_{\mathrm{Zar}}, (\mathcal{M}(r-1)/m)|_U)[-1] & \longrightarrow & \mathbb{R}\epsilon_* \underline{\mathbb{R}\mathrm{Hom}}(\mathbb{Z}[Z_U]_{\acute{e}t}, \mu_m^{\otimes(r-1)})[-1],
\end{array}$$

where the first vertical row is the exact triangle from above, the second vertical row is the corresponding exact triangle for étale sheaves and where the horizontal maps are induced by the cycle class maps, commutes.

*Proof.* Let  $A \rightarrow A''$  be the map of function algebras corresponding to  $i$  and  $A \rightarrow A'$  the map corresponding to the open inclusion  $V \rightarrow X$ . We let  $J$  be the category which is defined by gluing the object  $(0,0)$  of  $[1]^2$  to the object  $0$  of  $[1]$ . We call  $c$  the object  $1$  of  $[1]$  viewed as object of  $J$ , the other objects are numbered  $(k,l)$ ,  $k, l \in \{0,1\}$ . Let  $\mu_m^{\otimes r-1} \rightarrow \tilde{\mathcal{G}}$  be an injectively fibrant replacement in  $\mathrm{Cpx}(\mathrm{Sh}(\mathrm{Sm}_{U,\acute{e}t}, \mathbb{Z}/m))$ . We let  $\mathcal{H}_t(n)$  be the colimit of the  $\mathcal{G}^W((\mathrm{Spec} C_t^A) \times \Delta^n)$ , where  $W$  runs through the finite unions of elements of  $c^{r-1}(\mathrm{Spec} C_t^{A'}, n)$ . We denote by  $\mathcal{H}_t$  the total complex associated to the double complex which is the normalized complex associated to  $\tau_{\leq 2r} \mathcal{H}_t(\bullet)$ . We have an absolute purity isomorphism  $\varphi$  from the sheaf  $\mathrm{Sm}_U \ni Y \mapsto \ker(\mathcal{G}(Y \times_S X) \rightarrow \mathcal{G}(Y \times_X V))$  to  $\mathrm{Sm}_U \ni Y \mapsto \tilde{\mathcal{G}}(Y \times_S Z)[-2]$  in  $\mathrm{D}(\mathrm{Sh}(\mathrm{Sm}_{U,\acute{e}t}, \mathbb{Z}/m))$ . This can be lifted to a map of (complexes of) sheaves since the target of the map is injectively fibrant. We denote any such lift also by  $\varphi$ . We let  $\tilde{\mathcal{G}}^{r-1}$  and  $\tilde{\mathcal{G}}'$  be the analogues of  $\mathcal{G}^r$  and  $\mathcal{G}'$  (and in the first case also with the cycle conditions).

For  $K$  a  $n$ -simplex in the nerve of  $\mathcal{S}_p$  we define, using  $\varphi$ , a functor

$$\alpha_K^\heartsuit: \mathcal{E}_n \times I \times J \rightarrow \mathrm{Cpx}(\mathbf{Ab})$$

by sending

$$\begin{aligned}
(t, 0, c) & \text{ to } z_{F_t^{A''}}^{r-1}(C_t^{A''})[-2r], \quad (t, 0, (0,0)) \text{ to } z_{F_t^{A''}}^{r-1}(C_t^{A''})[-2r], \\
(t, 0, (1,0)) & \text{ to } z_{F_t^A}^r(C_t^A)[-2r], \quad (t, 0, (1,1)) \text{ to } z_{F_t^{A'}}^r(C_t^{A'})[-2r], \quad (t, 0, (0,1)) \text{ to } 0,
\end{aligned}$$

$(t, 1, c)$  to  $\tilde{\mathcal{G}}_{F_t^{A''}}^{r-1}(C_t^{A''})[-2]$ ,  $(t, 1, (0, 0))$  to  $\mathcal{H}_t$ ,  $(t, 1, (1, 0))$  to  $\mathcal{G}_{F_t^A}^r(C_t^A)$ ,  $(t, 1, (1, 1))$  to  $\mathcal{G}_{F_t^{A'}}^r(C_t^{A'})$ ,  $(t, 1, (0, 1))$  to 0,  
 $(t, 2, c)$  to  $\tilde{\mathcal{G}}'(C_t^{A''})[-2]$ ,  $(t, 2, (0, 0))$  to  $\ker(\mathcal{G}'(C_t^A) \rightarrow \mathcal{G}'(C_t^{A'}))$ ,  
 $(t, 2, (1, 0))$  to  $\mathcal{G}'(C_t^A)$ ,  $(t, 2, (1, 1))$  to  $\mathcal{G}'(C_t^{A'})$ ,  $(t, 2, (0, 1))$  to 0,  
 $(t, 3, c)$  to  $\tilde{\mathcal{G}}(C_t^{A''})[-2]$ ,  $(t, 3, (0, 0))$  to  $\ker(\mathcal{G}(C_t^A) \rightarrow \mathcal{G}(C_t^{A'}))$ ,  $(t, 3, (1, 0))$  to  $\mathcal{G}(C_t^A)$ ,  
 $(t, 3, (1, 1))$  to  $\mathcal{G}(C_t^{A'})$ ,  $(t, 3, (0, 1))$  to 0,  
 $(t, 4, c)$  to  $\tilde{\mathcal{G}}(A'' \otimes_D A_{\varphi_n(t)})[-2]$ ,  $(t, 4, (0, 0))$  to  $\ker(\mathcal{G}(A \otimes_D A_{\varphi_n(t)}) \rightarrow \mathcal{G}(A' \otimes_D A_{\varphi_n(t)}))$ ,  $(t, 4, (1, 0))$  to  $\mathcal{G}(A \otimes_D A_{\varphi_n(t)})$ ,  $(t, 4, (1, 1))$  to  $\mathcal{G}(A' \otimes_D A_{\varphi_n(t)})$  and  $(t, 4, (0, 1))$  to 0.

Sheafifying we obtain a functor  $\tilde{\alpha}_K^\heartsuit: \mathcal{E}_n \times I \times J \rightarrow \text{Cpx}(\text{Sh}(S_{Zar}, \mathbb{Z}))$ . For  $K$  a  $n$ -simplex in the nerve of  $\hat{\mathcal{S}}_p \times I \times J$  we let  $\gamma_K^\heartsuit := L^H(\Gamma \circ T^p(\tilde{\alpha}^\heartsuit)_K) \circ q_n$ , where  $p$  is the functor  $\hat{\mathcal{S}}_p \times I \times J \rightarrow \mathcal{S}_p \times I \times J$ .

The  $\gamma_K^\heartsuit$  glue to give a map

$$\gamma^\heartsuit: \mathcal{D}_{Q_\bullet}^{\hat{\mathcal{S}}_p \times I \times J} \rightarrow H_{qi}^H \text{Cpx}(\text{Ab}).$$

We denote by  $\gamma_\heartsuit' \in \text{Ho}(\text{Cpx}(\text{Ab})^{\hat{\mathcal{S}}_p \times I \times J})$  the diagram canonically associated to  $\gamma^\heartsuit$ .

The commutativity of the push forward of the corresponding  $I \times J$ -diagram in  $\text{Ho}(\text{Cpx}(\text{Ab})^{\hat{\mathcal{S}}_p})$  to  $\text{D}(\text{Sh}(\text{Sm}_{U, Zar}, \mathbb{Z}))$  shows the claim, using Corollary 5.27.  $\square$

### 5.3 The naive $\mathbb{G}_m$ -spectrum

**Proposition 5.29:** *There is a canonical isomorphism*

$$\mathcal{M}(r-1)[-1] \cong \underline{\mathbb{R}\text{Hom}}(\mathbb{Z}[\mathbb{G}_{m,S}, \{1\}]_{Zar}, \mathcal{M}(r))$$

in  $\text{D}(\text{Sh}(\text{Sm}_{S, Zar}, \mathbb{Z}))$ .

*Proof.* By Proposition 5.25 there is an exact triangle

$$\mathcal{M}(r-1)[-2] \rightarrow \underline{\mathbb{R}\text{Hom}}(\mathbb{Z}[\mathbb{A}_S^1]_{Zar}, \mathcal{M}(r)) \rightarrow \underline{\mathbb{R}\text{Hom}}(\mathbb{Z}[\mathbb{G}_{m,S}]_{Zar}, \mathcal{M}(r)) \rightarrow \mathcal{M}(r-1)[-1].$$

There is a split  $\underline{\mathbb{R}\text{Hom}}(\mathbb{Z}[\mathbb{G}_{m,S}]_{Zar}, \mathcal{M}(r)) \rightarrow \underline{\mathbb{R}\text{Hom}}(\mathbb{Z}[\mathbb{A}_S^1]_{Zar}, \mathcal{M}(r))$  induced by  $\{1\} \subset \mathbb{G}_{m,S}$  and the  $\mathbb{A}^1$ -invariance of  $\mathcal{M}(r)$ . This induces the required isomorphism.  $\square$

We thus get a naive  $\mathbb{Z}[\mathbb{G}_{m,S}, \{1\}]_{Zar}$ -spectrum  $\mathcal{M}$  (in the sense of [16, I 6]) in  $\text{D}(\text{Sh}(\text{Sm}_{S, Zar}, \mathbb{Z}))$  with entry  $\mathcal{M}(r)[r]$  in level  $r$ . We also denote a lift of  $\mathcal{M}$  to the homotopy category of  $\mathbb{Z}[\mathbb{G}_{m,S}, \{1\}]_{Zar}$ -spectra by  $\mathcal{M}$ .

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